For almost a decade, we have been receiving tremendous responses from GATE aspirants for our earlier books: GATE Multiple Choice Questions, GATE Guide, and the GATE Cloud series. Our first book, GATE Multiple Choice Questions (MCQ), was a compilation of objective questions and solutions for all subjects of GATE Electronics & Communication Engineering in one book. The idea behind the book was that Gate aspirants who had just completed or about to finish their last semester to achieve his or her B.E/B.Tech need only to practice answering questions to crack GATE. The solutions in the book were presented in such a manner that a student needs to know fundamental concepts to understand them. We assumed that students have learned enough of the fundamentals by his or her graduation. The book was a great success, but still there were a large ratio of aspirants who needed more preparatory materials beyond just problems and solutions. This large ratio mainly included average students.

Later, we perceived that many aspirants couldn’t develop a good problem solving approach in their B.E/B.Tech. Some of them lacked the fundamentals of a subject and had difficulty understanding simple solutions. Now, we have an idea to enhance our content and present two separate books for each subject: one for theory, which contains brief theory, problem solving methods, fundamental concepts, and points-to-remember. The second book is about problems, including a vast collection of problems with descriptive and step-by-step solutions that can be understood by an average student. This was the origin of GATE Guide (the theory book) and GATE Cloud (the problem bank) series: two books for each subject. GATE Guide and GATE Cloud were published in three subjects only.

Thereafter we received an immense number of emails from our readers looking for a complete study package for all subjects and a book that combines both GATE Guide and GATE Cloud. This encouraged us to present GATE Study Package (a set of 10 books: one for each subject) for GATE Electronic and Communication Engineering. Each book in this package is adequate for the purpose of qualifying GATE for an average student. Each book contains brief theory, fundamental concepts, problem solving methodology, summary of formulae, and a solved question bank. The question bank has three exercises for each chapter: 1) Theoretical MCQs, 2) Numerical MCQs, and 3) Numerical Type Questions (based on the new GATE pattern). Solutions are presented in a descriptive and step-by-step manner, which are easy to understand for all aspirants.

We believe that each book of GATE Study Package helps a student learn fundamental concepts and develop problem solving skills for a subject, which are key essentials to crack GATE. Although we have put a vigorous effort in preparing this book, some errors may have crept in. We shall appreciate and greatly acknowledge all constructive comments, criticisms, and suggestions from the users of this book. You may write to us at rajkumar. kanodia@gmail.com and ashish.murolia@gmail.com.

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We wish you good luck!
R. K. Kanodia
Ashish Murolia
GATE Electronics & Communications:

IES Electronics & Telecommunication
Classification of signals and systems: System modelling in terms of differential and difference equations; State variable representation; Fourier series; Fourier transforms and their application to system analysis; Laplace transforms and their application to system analysis; Convolution and superposition integrals and their applications; Z-transforms and their applications to the analysis and characterisation of discrete time systems; Random signals and probability, Correlation functions; Spectral density; Response of linear system to random inputs.

**********
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6.1 INTRODUCTION

As we studied in previous chapter, the Laplace transform is an important tool for analysis of continuous time signals and systems. Similarly, z-transforms enables us to analyze discrete time signals and systems in the z-domain.

Like, the Laplace transform, it is also classified as bilateral z-transform and unilateral z-transform.

The bilateral or two-sided z-transform is used to analyze both causal and non-causal LTI discrete systems, while the unilateral z-transform is defined only for causal signals.

NOTE:
The properties of z-transform are similar to those of the Laplace transform.

6.1.1 The Bilateral or Two-Sided z-transform

The z-transform of a discrete-time sequence \( x[n] \), is defined as

\[
X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}
\]  

(6.1.1)

Where, \( X(z) \) is the transformed signal and \( \mathcal{Z} \) represents the z-transformation. \( z \) is a complex variable. In polar form, \( z \) can be expressed as

\[ z = re^{j\Omega} \]

where \( r \) is the magnitude of \( z \) and \( \Omega \) is the angle of \( z \). This corresponds to a circle in \( z \)-plane with radius \( r \) as shown in figure 6.1.1 below.

![z-plane diagram](image)

Figure 6.1.1 z-plane

NOTE:
The signal \( x[n] \) and its z-transform \( X(z) \) are said to form a z-transform pair denoted as

\[ x[n] \xrightarrow{\mathcal{Z}} X(z) \]
6.1.2 The Unilateral or One-sided $z$-Transform

The $z$-transform for causal signals and systems is referred to as the unilateral $z$-transform. For a causal sequence $x[n] = 0$, for $n < 0$

Therefore, the unilateral $z$-transform is defined as

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

(6.1.2)

**NOTE:**
For causal signals and systems, the unilateral and bilateral $z$-transform are the same.

6.2 Existence of $z$-Transform

Consider the bilateral $z$-transform given by equation (6.1.1)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

The $z$-transform exists when the infinite sum in above equation converges. For this summation to be converged, $|x[n] z^{-n}|$ must be absolutely summable.

Substituting $z = re^{j\omega}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] (re^{j\omega})^{-n}$$

or,

$$X(z) = \sum_{n=-\infty}^{\infty} \{x[n] r^{-n}\} e^{-j\omega n}$$

Thus for existence of $z$-transform

$$|X(z)| < \infty \quad \sum_{n=-\infty}^{\infty} |x[n]| r^{-n} < \infty$$

(6.2.1)

6.3 Region of Convergence

The existence of $z$-transform is given from equation (6.2.1). The values of $r$ for which $x[n] r^{-n}$ is absolutely summable is referred to as region of convergence. Since, $z = re^{j\omega}$ so $r = |z|$. Therefore we conclude that the range of values of the variable $|z|$ for which the sum in equation (6.1.1) converges is called the region of convergence. This can be explained through the following examples.

6.3.1 Poles and Zeros of Rational $z$-transforms

The most common form of $z$-transform is a rational function. Let $X(z)$ be the $z$-transform of sequence $x[n]$, expressed as a ratio of two polynomials $N(z)$ and $D(z)$.

$$X(z) = \frac{N(z)}{D(z)}$$

The roots of numerator polynomial i.e. values of $z$ for which $X(z) = 0$ is referred to as zeros of $X(z)$. The roots of denominator polynomial for which $X(z) = \infty$ is referred to as poles of $X(z)$. The representation of $X(z)$ through its poles and zeros in the $z$-plane is called pole-zero plot of $X(z)$.

For example consider a rational transfer function $X(z)$ given as

$$H(z) = \frac{z}{z^2 - 5z + 6} = \frac{z}{(z - 2)(z - 3)}$$
Now, the zeros of \( X(z) \) are roots of numerator that is \( z = 0 \) and poles are roots of equation \((z - 2)(z - 3) = 0\) which are given as \( z = 2 \) and \( z = 3 \). The poles and zeros of \( X(z) \) are shown in pole-zero plot of figure 6.3.1.

![Pole-zero plot of \( X(z) \)](image)

**NOTE:**
In pole-zero plot poles are marked by a small cross ‘\( \times \)’ and zeros are marked by a small dot ‘\( o \)’ as shown in figure 6.3.1.

### 6.3.2 Properties of ROC

The various properties of ROC are summarized as follows. These properties can be proved by taking appropriate examples of different DT signals.

**PROPERTY 1**

The ROC is a concentric ring in the \( z \)-plane centered about the origin.

**PROOF:**

The \( z \)-transform is defined as

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}
\]

Put \( z = re^{j\omega} \)

\[
X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n}
\]

\( X(z) \) converges for those values of \( z \) for which \( x[n]r^{-n} \) is absolutely summable that is

\[
\sum_{n=-\infty}^{\infty} x[n]r^{-n} < \infty
\]

Thus, convergence is dependent only on \( r \), where, \( r = |z| \)

The equation \( z = re^{j\omega} \), describes a circle in \( z \)-plane. Hence the ROC will consists of concentric rings centered at zero.

**PROPERTY 2**

The ROC cannot contain any poles.

**PROOF:**

ROC is defined as the values of \( z \) for which \( z \)-transform \( X(z) \) converges. We know that \( X(z) \) will be infinite at pole, and, therefore \( X(z) \) does not converge at poles. Hence the region of convergence does not include any pole.
**PROPERTY 3**

If \( x[n] \) is a finite duration two-sided sequence then the ROC is entire \( z \)-plane except at \( z = 0 \) and \( z = \infty \).

**PROOF :**

A sequence which is zero outside a finite interval of time is called ‘finite duration sequence’. Consider a finite duration sequence \( x[n] \) shown in figure 6.3.2a; \( x[n] \) is non-zero only for some interval \( N_1 \leq n \leq N_2 \).

![Figure 6.3.2a A Finite Duration Sequence](image)

The \( z \)-transform of \( x[n] \) is defined as

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}
\]

This summation converges for all finite values of \( z \). If \( N_1 \) is negative and \( N_2 \) is positive, then \( X(z) \) will have both positive and negative powers of \( z \). The negative powers of \( z \) becomes unbounded (infinity) if \( |z| \to 0 \). Similarly positive powers of \( z \) becomes unbounded (infinity) if \( |z| \to \infty \). So ROC of \( X(z) \) is entire \( z \)-plane except possible \( z = 0 \) and/or \( z = \infty \).

**NOTE :**

Both \( N_1 \) and \( N_2 \) can be either positive or negative.

**PROPERTY 4**

If \( x[n] \) is a right-sided sequence, and if the circle \( |z| = r_0 \) is in the ROC, then all values of \( z \) for which \( |z| > r_0 \) will also be in the ROC.

**PROOF :**

A sequence which is zero prior to some finite time is called the right-sided sequence. Consider a right-sided sequence \( x[n] \) shown in figure 6.3.2b; that is; \( x[n] = 0 \) for \( n < N_1 \).

![Figure 6.3.2b A Right - Sided Sequence](image)
Let the z-transform of $x[n]$ converges for some value of $|z| = r_0$. From the condition of convergence we can write

$$\left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| < \infty$$

$$\sum_{n=-\infty}^{\infty} |x[n]|r_0^{-n} < \infty$$

The sequence is right sided, so limits of above summation changes as

$$\sum_{n=-N_1}^{\infty} |x[n]|r_0^{-n} < \infty \quad (6.3.1)$$

Now if we take another value of $z$ as $|z| = r_1$ with $r_1 < r_0$, then $x[n]r_1^{-n}$ decays faster than $x[n]r_0^{-n}$ for increasing $n$. Thus we can write

$$\sum_{n=-N_1}^{\infty} |x[n]|z^{-n} = \sum_{n=-N_1}^{\infty} |x[n]|z^{-n}r_0^{-n}r_1^n$$

$$= \sum_{n=-N_1}^{\infty} |x[n]|r_0^{-n}(\frac{z}{r_0})^{-n} \quad (6.3.2)$$

From equation (6.3.1) we know that $x[n]r_0^{-n}$ is absolutely summable. Let, it is bounded by some value $M_x$, then equation (6.3.2) becomes as

$$\sum_{n=-N_1}^{\infty} |x[n]|z^{-n} \leq M_x \sum_{n=-N_1}^{\infty} (\frac{z}{r_0})^{-n} \quad (6.3.3)$$

The right hand side of above equation converges only if

$$|\frac{z}{r_0}| > 1 \text{ or } |z| > r_0$$

Thus, we conclude that if the circle $|z| = r_0$ is in the ROC, then all values of $z$ for which $|z| > r_0$ will also be in the ROC. The ROC of a right-sided sequence is illustrated in figure 6.3.2c.

**PROPERTY 5**

If $x[n]$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of $z$ for which $|z| < r_0$ will also be in the ROC.

**PROOF:**

A sequence which is zero after some finite time interval is called a ‘left-sided signal’. Consider a left-sided signal $x[n]$ shown in figure 6.3.2d; that is $x[n] = 0$ for $n > N_2$. 

![Figure 6.3.2c ROC of a right-sided sequence](image-url)
Let $z$-transform of $x[n]$ converges for some values of $|z|$ (i.e. $|z| = r_0$).

From the condition of convergence we write

$$\left| \sum_{n=-\infty}^{\infty} x[n]z^n \right| < \infty$$

or

$$\sum_{n=-\infty}^{\infty} |x[n]| r_0^n < \infty$$

(6.3.4)

The sequence is left sided, so the limits of summation changes as

$$\sum_{n=-\infty}^{N} |x[n]| r_0^n < \infty$$

(6.3.5)

Now if take another value of $z$ as $|z| = r_1$, then we can write

$$\sum_{n=-\infty}^{N} |x[n]| z^n = \sum_{n=-\infty}^{N} |x[n]| r_0^n r_1^{-n}$$

$$= \sum_{n=-\infty}^{N} |x[n]| r_0^n \left(\frac{r_0}{z}\right)^n$$

(6.3.6)

From equation (6.3.4), we know that $x[n] r_0^{-n}$ is absolutely summable. Let it is bounded by some value $M_x$, then equation (6.3.6) becomes as

$$\sum_{n=-\infty}^{N} |x[n]| z^n \leq M_x \sum_{n=-\infty}^{N} \left(\frac{r_0}{z}\right)^n$$

The above summation converges if $\left|\frac{r_0}{z}\right| > 1$ (because $n$ is increasing negatively), so $|z| < r_0$ will be in ROC.

The ROC of a left-sided sequence is illustrated in figure 6.3.2e.

**PROPERTY 6**

If $x[n]$ is a two-sided signal, and if the circle $|z| = r_0$ is in the ROC, then the ROC consists of a ring in the $z$-plane that includes the circle $|z| = r_0$. 

![Figure 6.3.2d](image-url) A left-sided sequence

![Figure 6.3.2e](image-url) ROC of a Left-Sided Sequence

**Figure 6.3.2d** A left-sided sequence

**Figure 6.3.2e** ROC of a Left-Sided Sequence
PROOF:
A sequence which is defined for infinite extent for both $n > 0$ and $n < 0$ is called ‘two-sided sequence’. A two-sided signal $x[n]$ is shown in figure 6.3.2f.

Figure 6.3.2f A Two-Sided Sequence

For any time $N_0$, a two-sided sequence can be divided into sum of left-sided and right-sided sequences as shown in figure 6.3.2g.

Figure 6.3.2g A Two-Sided Sequence Divided into Sum of a Left-Sided and Right-Sided Sequence

The $z$-transform of $x[n]$ converges for the values of $z$ for which the transform of both $x[n]$ and $x[n]$ converges. From property 4, the ROC of a right-sided sequence is a region which is bounded on the inside by a circle and extending outward to infinity i.e. $|z| > r_1$. From property 5, the ROC of a left-sided sequence is bounded on the outside by a circle and extending inward to zero i.e. $|z| < r_2$. So the ROC of combined signal includes intersection of both ROCs which is ring in the $z$-plane.

The ROC for the right-sided sequence $x[n]$, the left-sequence $x[n]$ and their combination which is a two-sided sequence $x[n]$ are shown in figure 6.3.2h.
PROPERTY 7
If the z-transform $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or extends to infinity.

PROOF:
The exponential DT signals also have rational z-transform and the poles of $X(z)$ determines the boundaries of ROC.

PROPERTY 8
If the z-transform $X(z)$ of $x[n]$ is rational and $x[n]$ is a right-sided sequence then the ROC is the region in the $z$-plane outside the outermost pole i.e. ROC is the region outside a circle with a radius greater than the magnitude of largest pole of $X(z)$.

PROOF:
This property can be be proved by taking property 4 and 7 together.

PROPERTY 9
If the z-transform $X(z)$ of $x[n]$ is rational and $x[n]$ is a left-sided sequence then the ROC is the region in the $z$-plane inside the innermost pole i.e. ROC is the region inside a circle with a radius equal to the smallest magnitude of poles of $X(z)$.

PROOF:
This property can be be proved by taking property 5 and 7 together.

-Transform of Some Basic Functions
Z-transform of basic functions are summarized in the Table 6.1 with their respective ROCs.

6.4 THE INVERSE -TRANSFORM
Let $X(z)$ be the z-transform of a sequence $x[n]$. To obtain the sequence $x[n]$ from its z-transform is called the inverse z-transform. The inverse z-transform is given as
<table>
<thead>
<tr>
<th>DT sequence $x[n]$</th>
<th>$z$-transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\delta[n]$</td>
<td>1</td>
<td>entire $z$-plane</td>
</tr>
<tr>
<td>2. $\delta[n - n_0]$</td>
<td>$z^{-n_0}$</td>
<td>entire $z$-plane, except $z = 0$</td>
</tr>
<tr>
<td>3. $u[n]$</td>
<td>$\frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$</td>
<td>$</td>
</tr>
<tr>
<td>4. $\alpha^n u[n]$</td>
<td>$\frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}$</td>
<td>$</td>
</tr>
<tr>
<td>5. $\alpha^{-1}u[n - 1]$</td>
<td>$\frac{z^{-1}}{1 - \alpha z^{-1}} = \frac{1}{z - \alpha}$</td>
<td>$</td>
</tr>
<tr>
<td>6. $nu[n]$</td>
<td>$\frac{z^{-1}}{(1 - z^{-1})^2} = \frac{z}{(z - 1)^2}$</td>
<td>$</td>
</tr>
<tr>
<td>7. $n\alpha^n u[n]$</td>
<td>$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2} = \frac{\alpha z}{(z - \alpha)^2}$</td>
<td>$</td>
</tr>
<tr>
<td>8. $\cos(\Omega_0 n) u[n]$</td>
<td>$\frac{1 - z^{-1}\cos \Omega_0}{1 - 2z^{-1}\cos \Omega_0 + z^{-2}}$ or $\frac{z[z - \cos \Omega_0]}{z^2 - 2z\cos \Omega_0 + 1}$</td>
<td>$</td>
</tr>
<tr>
<td>9. $\sin(\Omega_0 n) u[n]$</td>
<td>$\frac{z^{-1}\sin \Omega_0}{1 - 2z^{-1}\cos \Omega_0 + z^{-2}}$ or $\frac{z\sin \Omega_0}{z^2 - 2z\cos \Omega_0 + 1}$</td>
<td>$</td>
</tr>
<tr>
<td>10. $\alpha^n \cos(\Omega_0 n) u[n]$</td>
<td>$\frac{1 - \alpha z^{-1}\cos \Omega_0}{1 - 2\alpha z^{-1}\cos \Omega_0 + \alpha^2 z^{-2}}$ or $\frac{z[z - \alpha \cos \Omega_0]}{z^2 - 2\alpha z\cos \Omega_0 + \alpha^2}$</td>
<td>$</td>
</tr>
<tr>
<td>11. $\alpha^n \sin(\Omega_0 n) u[n]$</td>
<td>$\frac{\alpha z^{-1}\sin \Omega_0}{1 - 2\alpha z^{-1}\cos \Omega_0 + \alpha^2 z^{-2}}$ or $\frac{\alpha z\sin \Omega_0}{z^2 - 2\alpha z\cos \Omega_0 + \alpha^2}$</td>
<td>$</td>
</tr>
<tr>
<td>12. $r\alpha^n \sin(\Omega_0 n + \theta) u[n]$ with $\alpha \in \mathbb{R}$</td>
<td>$\frac{A + Bz^{-1}}{1 + 2\gamma z^{-1} + \alpha^2 z^{-2}}$ or $\frac{z(Az + B)}{z^2 + 2\gamma z + \gamma^2}$</td>
<td>$</td>
</tr>
</tbody>
</table>
\[ x[n] = \frac{1}{2\pi j} \int X(z) z^{-1} dz \]

This method involves the contour integration, so difficult to solve. There are other commonly used methods to evaluate the inverse z-transform given as follows
1. Partial fraction method
2. Power series expansion

6.4.1 Partial Fraction Method

If \( X(z) \) is a rational function of \( z \) then it can be expressed as follows.

\[ X(z) = \frac{N(z)}{D(z)} \]

It is convenient if we consider \( X(z)/z \) rather than \( X(z) \) to obtain the inverse z-transform by partial fraction method.

Let \( p_1, p_2, p_3, \ldots, p_n \) are the roots of denominator polynomial, also the poles of \( X(z) \). Then, using partial fraction method \( X(z)/z \) can be expressed as

\[ \frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \frac{A_3}{z - p_3} + \ldots + \frac{A_n}{z - p_n} \]

\[ X(z) = A_1 \frac{1}{z - p_1} + A_2 \frac{1}{z - p_2} + \ldots + A_n \frac{1}{z - p_n} \]

Now, the inverse z-transform of above equation can be obtained by comparing each term with the standard z-transform pair given in table 6.1. The values of coefficients \( A_1, A_2, A_3, \ldots, A_n \) depends on whether the poles are real & distinct or repeated or complex. Three cases are given as follows

Case I : Poles are Simple and Real

\( X(z)/z \) can be expanded in partial fraction as

\[ \frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \frac{A_3}{z - p_3} + \ldots + \frac{A_n}{z - p_n} \] (6.4.1)

where \( A_1, A_2, \ldots, A_n \) are calculated as follows

\[ A_1 = \left( z - p_1 \right) \left. \frac{X(z)}{z} \right|_{z = p_1} \]

\[ A_2 = \left( z - p_2 \right) \left. \frac{X(z)}{z} \right|_{z = p_2} \]

In general,

\[ A_1 = \left( z - p_i \right) \left. \frac{X(z)}{z} \right|_{z = p_i} \] (6.4.2)

Case II : If Poles are Repeated

In this case \( X(z)/z \) has a different form. Let \( p_k \) be the root which repeats \( l \) times, then the expansion of equation must include terms

\[ \frac{X(z)}{z} = \frac{A_{nk}}{z - p_k} + \frac{A_{2k}}{(z - p_k)^2} + \ldots + \frac{A_{lk}}{(z - p_k)^l} \] (6.4.3)

The coefficient \( A_{ik} \) are evaluated by multiplying both sides of equation (6.4.3) by \( (z - p_k)^l \), differentiating \((l - 1)\) times and then evaluating the resultant equation at \( z = p_k \).

Thus,

\[ C_{ik} = \frac{1}{l!} \left. \frac{d^{l-1}}{dz^{l-1}} \left( z - p_k \right)^l \left. \frac{X(z)}{z} \right|_{z = p_k} \right|_{z = p_k} \] (6.4.4)
Case III : Complex Poles

If \( X(z) \) has complex poles then partial fraction of the \( X(z)/z \) can be expressed as

\[
\frac{X(z)}{z} = \frac{A_1}{z - p_1^*} + \frac{A_1^*}{z - p_1}
\]

(6.4.5)

where \( A_1^* \) is complex conjugate of \( A_1 \) and \( p_1^* \) is complex conjugate of \( z_1 \). The coefficients are obtained by equation (6.4.2)

6.4.2 Power Series Expansion Method

Power series method is also convenient in calculating the inverse \( z \)-transform. The \( z \)-transform of sequence \( x[n] \) is given as

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}
\]

Now, \( X(z) \) is expanded in the following form

\[
X(z) = \ldots + x[-2]z^2 + x[-1]z^1 + x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots
\]

To obtain inverse \( z \)-transform (i.e. \( x[n] \)), represent the given \( X(z) \) in the form of above power series. Then by comparing we can get

\[
x[n] = \{\ldots x[-2], x[-1], x[0], x[1], x[2], \ldots\}
\]

6.5 PROPERTIES OF \( z \)-TRANSFORM

The unilateral and bilateral \( z \)-transforms possess a set of properties, which are useful in the analysis of DT signals and systems. The proofs of properties are given for bilateral transform only and can be obtained in a similar way for the unilateral transform.

6.5.1 Linearity

Like Laplace transform, the linearity property of \( z \) transform states that, the linear combination of DT sequences in the time domain is equivalent to linear combination of their \( z \) transform.

Let

\[
\begin{align*}
\mathcal{Z} x_1[n] & \Longleftrightarrow X_1(z), & \text{with ROC: } R_1 \\
\mathcal{Z} x_2[n] & \Longleftrightarrow X_2(z), & \text{with ROC: } R_2
\end{align*}
\]

then,

\[
\mathcal{Z} ax_1[n] + bx_2[n] \Longleftrightarrow aX_1(z) + bX_2(z),
\]

with ROC: at least \( R_1 \cap R_2 \)

for both unilateral and bilateral \( z \)-transform.

PROOF:

The \( z \)-transform of signal \( \{ax_1[n] + bx_2[n]\} \) is given by equation (6.1.1) as follows

\[
\mathcal{Z} \{ax_1[n] + bx_2[n]\} = \sum_{n=-\infty}^{\infty} (ax_1[n] + bx_2[n])z^{-n}
\]

\[
= a \sum_{n=-\infty}^{\infty} x_1[n]z^{-n} + b \sum_{n=-\infty}^{\infty} x_2[n]z^{-n}
\]

\[
= aX_1(z) + bX_2(z)
\]

Hence, \( ax_1[n] + bx_2[n] \Longleftrightarrow aX_1(z) + bX_2(z) \)

ROC : Since, the \( z \)-transform \( X_1(z) \) is finite within the specified ROC, \( R_1 \).
Similarly, $X_2(z)$ is finite within its ROC, $R_2$. Therefore, the linear combination $aX_1(z) + bX_2(z)$ should be finite at least within region $R_2 \cap R_2$.

**NOTE:**
In certain cases, due to the interaction between $x_1[n]$ and $x_2[n]$, which may lead to cancellation of certain terms, the overall ROC may be larger than the intersection of the two regions. On the other hand, if there is no common region between $R_1$ and $R_2$, the z-transform of $ax_1[n] + bx_2[n]$ does not exist.

### 6.5.2 Time Shifting

#### For the bilateral z-transform

If

$$x[n] \leftrightarrow Z \rightarrow X(z), \quad \text{with ROC } R_x$$

then

$$x[n - n_0] \leftrightarrow Z \rightarrow z^{-n_0}X(z),$$

and

$$x[n + n_0] \leftrightarrow Z \rightarrow z^{n_0}X(z),$$

with ROC : $R_x$ except for the possible deletion or addition of $z = 0$ or $z = \infty$.

**PROOF:**

The bilateral z-transform of signal $x[n - n_0]$ is given by equation (6.1.1) as follows

$$Z \{x[n - n_0]\} = \sum_{n=-\infty}^{\infty} x[n - n_0]z^{-n}$$

Substituting $n - n_0 = \alpha$ on RHS, we get

$$Z \{x[n - n_0]\} = \sum_{\alpha=-\infty}^{\infty} x[\alpha]z^{-(n_0 + \alpha)}$$

$$= \sum_{\alpha=-\infty}^{\infty} x[\alpha]z^{-n_0}z^{-\alpha} = z^{-n_0} \sum_{\alpha=-\infty}^{\infty} x[\alpha]z^{-\alpha}$$

$$Z \{x[n - n_0]\} = z^{-n_0}X(z)$$

Similarly we can prove

$$Z \{x[n + n_0]\} = z^{n_0}X(z)$$

**ROC** : The ROC of shifted signals is altered because of the terms $z^{n_0}$ or $z^{-n_0}$, which affects the roots of the denominator in $X(z)$.

#### TIME SHIFTING FOR UNILATERAL Z-TRANSFORM

For the unilateral z-transform

If

$$x[n] \leftrightarrow Z \rightarrow X(z), \quad \text{with ROC } R_x$$

then

$$x[n - n_0] \leftrightarrow Z \rightarrow z^{-n_0}(X(z) + \sum_{m=1}^{n_0} x[-m]z^{n-m}),$$

and

$$x[n + n_0] \leftrightarrow Z \rightarrow z^{n_0}(X(z) - \sum_{m=0}^{n_0-1} x[m]z^{-m}),$$

with ROC : $R_x$ except for the possible deletion or addition of $z = 0$ or $z = \infty$.

**PROOF:**

The unilateral z-transform of signal $x[n - n_0]$ is given by equation (6.1.2) as follows
The Z-Transform

\[ \mathcal{Z} \{ x[n - n_0] \} = \sum_{n=0}^{\infty} x[n - n_0] z^{-n} \]

Multiplying RHS by \( z^{n_0} \) and \( z^{-n_0} \)

\[ \mathcal{Z} \{ x[n - n_0] \} = \sum_{n=0}^{\infty} x[n - n_0] z^{-n} z^{n_0} z^{-n_0} \]

\[ = z^{-n_0} \sum_{n=0}^{\infty} x[n - n_0] z^{-(n-n_0)} \]

Substituting \( n - n_0 = \alpha \)

Limits; when \( n \to 0 \), \( \alpha \to -n_0 \)

when \( n \to +\infty \), \( \alpha \to +\infty \)

Now, \( \mathcal{Z} \{ x[n - n_0] \} = z^{-n_0} \sum_{\alpha=-n_0}^{\infty} x[\alpha] z^{-\alpha} \)

\[ = z^{-n_0} \sum_{\alpha=-n_0}^{\infty} x[\alpha] z^{-\alpha} + z^{-n_0} \sum_{\alpha=0}^{\infty} x[\alpha] z^{-\alpha} \]

or, \( \mathcal{Z} \{ x[n - n_0] \} = z^{-n_0} \sum_{\alpha=0}^{\infty} x[\alpha] z^{-\alpha} + z^{-n_0} \sum_{\alpha=1}^{\infty} x[-\alpha] z^{-\alpha} \)

or, \( \mathcal{Z} \{ x[n - n_0] \} = z^{-n_0} \sum_{\alpha=0}^{\infty} x[\alpha] z^{-\alpha} + z^{-n_0} \sum_{\alpha=1}^{\infty} x[-\alpha] z^{-\alpha} \)

Changing the variables as \( \alpha \to n \) and \( \alpha \to m \) in first and second summation respectively

\[ \mathcal{Z} \{ x[n - n_0] \} = z^{-n_0} \sum_{n=0}^{\infty} x[n] z^{-n} + z^{-n_0} \sum_{m=1}^{\infty} x[-m] z^{-m} \]

\[ = z^{-n_0} X[z] + z^{-n_0} \sum_{m=1}^{\infty} x[-m] z^{-m} \]

In similar way, we can also prove that

\[ x[n + n_0] \xrightarrow{z^{-n}} z^n \left( X(z) - \sum_{m=0}^{n-1} x[m] z^{-m} \right) \]

### 6.5.3 Time Reversal

Time reversal property states that time reflection of a DT sequence in time domain is equivalent to replacing \( z \) by \( 1/z \) in its \( z \)-transform.

**If**

\[ x[n] \xrightarrow{\mathcal{Z}} X(z), \text{ with ROC : } R_x \]

**then**

\[ x[-n] \xrightarrow{\mathcal{Z}} X \left( \frac{1}{z} \right), \text{ with ROC : } 1/R_x \]

for bilateral \( z \)-transform.

**PROOF:**

The bilateral \( z \)-transform of signal \( x[-n] \) is given by equation (6.1.1) as follows

\[ \mathcal{Z} \{ x[-n] \} = \sum_{n=-\infty}^{\infty} x[-n] z^{-n} \]

Substituting \( -n = k \) on the RHS, we get

\[ \mathcal{Z} \{ x[-n] \} = \sum_{k=-\infty}^{\infty} x[k] z^{k} = \sum_{k=-\infty}^{\infty} x[k] (z^{-1})^{-k} = X \left( \frac{1}{z} \right) \]

Hence,

\[ x[-n] \xrightarrow{\mathcal{Z}} X \left( \frac{1}{z} \right) \]

**ROC :** \( z^{-1} \in R_x \) or \( z \in 1/R_x \)

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6.5.4 Differentiation in the \( z \)-domain

This property states that multiplication of time sequence \( x[n] \) with \( n \) corresponds to differentiation with respect to \( z \) and multiplication of result by \(-z\) in the \( z\)-domain.

\[
\text{If } x[n] \xrightarrow{z} X(z), \quad \text{with ROC : } R_x
\]

\[
\text{then } nx[n] \xrightarrow{z} -z \frac{dX(z)}{dz}, \quad \text{with ROC : } R_x
\]

For both unilateral and bilateral \( z \)-transforms.

**Proof:**

The bilateral \( z \)-transform of signal \( x[n] \) is given by equation (6.1.1) as follows

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}
\]

Differentiating both sides with respect to \( z \) gives

\[
\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x[n]\frac{dz^{-n}}{dz} = \sum_{n=-\infty}^{\infty} x[n](-nz^{-n-1})
\]

Multiplying both sides by \(-z\), we obtain

\[
-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n}
\]

Hence,

\[
x[n] \xrightarrow{z} -z \frac{dX(z)}{dz}, \quad \text{with ROC : } R_x
\]

**ROC:** This operation does not affect the ROC.

6.5.5 Scaling in \( z \)-Domain

Multiplication of a time sequence with an exponential sequence \( a^n \) corresponds to scaling in \( z \)-domain by a factor of \( a \).

\[
\text{If } x[n] \xrightarrow{z} X(z), \quad \text{with ROC : } R_x
\]

\[
a^n x[n] \xrightarrow{z} X\left(\frac{z}{a}\right), \quad \text{with ROC : } |a| R_x
\]

For both unilateral and bilateral transforms.

**Proof:**

The bilateral \( z \)-transform of signal \( x[n] \) is given by equation (6.1.1) as

\[
\mathcal{Z}\{a^n x[n]\} = \sum_{n=-\infty}^{\infty} a^n x[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n][a^{-1}z]^{-n}
\]

\[
a^n x[n] \xrightarrow{z} X\left(\frac{z}{a}\right)
\]

**ROC:** If \( z \) is a point in the ROC of \( X(z) \) then the point \( |a|z \) is in the ROC of \( X(z/a) \).

6.5.6 Time Scaling

As we discussed in Chapter 2, there are two types of scaling in the DT domain decimation(compression) and interpolation(expansion).

**Time Compression**

Since the decimation (compression) of DT signals is an irreversible process (because some data may lost), therefore the \( z \)-transform of \( x[n] \) and its
decimated sequence \( y[n] = x[n] \) not be related to each other.

**Time Expansion**

In the discrete time domain, time expansion of sequence \( x[n] \) is defined as

\[
x_k[n] = \begin{cases} 
  x[n/k] & \text{if } n \text{ is a multiple of integer } k \\
  0 & \text{otherwise}
\end{cases}
\]

(6.5.1)

Time-scaling property of z-transform is derived only for time expansion which is given as

\[
\text{If } \quad x[n] \leftrightarrow X(z), \quad \text{with ROC : } R_x \\
\text{then } \quad x_k[n] \leftrightarrow X_k(z) = X(z^k), \quad \text{with ROC : } (R_x)^{1/k}
\]

for both the unilateral and bilateral z-transform.

**PROOF :**

The unilateral z-transform of expanded sequence \( x_k[n] \) is given by

\[
\mathcal{Z}\{x_k[n]\} = \sum_{n=0}^{\infty} x_k[n]z^{-n} = x_k[0] + x_k[1]z^{-1} + \ldots + x_k[k]z^{-k} + x_k[k+1]z^{-(k+1)} + \ldots + x_k[2k]z^{-2k} + \ldots
\]

Since the expanded sequence \( x_k[n] \) is zero everywhere except when \( n \) is a multiple of \( k \). This reduces the above transform as follows

\[
\mathcal{Z}\{x_k[n]\} = x_k[0] + x_k[k]z^{-k} + x_k[2k]z^{-2k} + x_k[3k]z^{-3k} + \ldots
\]

As defined in equation 6.5.1, interpolated sequence is

\[
x_k[n] = x[n/k]
\]

\[
n = 0 \quad x_k[0] = x[0].
\]

\[
n = k \quad x_k[k] = x[1]
\]

\[
n = 2k \quad x_k[2k] = x[2]
\]

Thus, we can write

\[
\mathcal{Z}\{x_k[n]\} = x[0] + x[1]z^{-k} + x[2]z^{-2k} + x[3]z^{-3k} + \ldots = \sum_{n=0}^{\infty} x[n](z^k)^{-n} = X(z^k)
\]

**NOTE :**

Time expansion of a DT sequence by a factor of \( k \) corresponds to replacing \( z \) as \( z^k \) in its z-transform.

### 6.5.7 Time Differencing

If \( x[n] \leftrightarrow X(z), \quad \text{with ROC : } R_x \)

then

\[
x[n] - x[n-1] \leftrightarrow (1 - z^{-1})X(z), \quad \text{with the ROC : } R_x \text{ except for the possible deletion of } z = 0, \text{ for both unilateral and bilateral transform.}
\]

**PROOF :**

The z-transform of \( x[n] - x[n-1] \) is given by equation (6.1.1) as follows

\[
\mathcal{Z}\{x[n] - x[n-1]\} = \sum_{n=-\infty}^{\infty} [x[n] - x[n-1]]z^{-n}
\]
The Z-Transform

6.5.8 Time Convolution

Time convolution property states that convolution of two sequence in time domain corresponds to multiplication in $z$-domain.

Let $x_1[n] \xrightarrow{z} X_1(z)$, ROC : $R_1$
and $x_2[n] \xrightarrow{z} X_2(z)$, ROC : $R_2$
then the convolution property states that
$x_1[n] * x_2[n] \xrightarrow{z} X_1(z) X_2(z)$, ROC : at least $R_1 \cap R_2$
for both unilateral and bilateral $z$-transforms.

PROOF :

As discussed in chapter 4, the convolution of two sequences is given by

$$x_1[n] * x_2[n] = \sum_{k=0}^{\infty} x_1[k] x_2[n-k]$$

Taking the $z$-transform of both sides gives

$$x_1[n] * x_2[n] \xrightarrow{z} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] z^{-n}$$

Interchanging the order of the two summations, we get

$$x_1[n] * x_2[n] \xrightarrow{z} \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k] z^{-n}$$

Substituting $n-k = \alpha$ in the second summation

$$x[n] * x_2[n] \xrightarrow{z} \sum_{k=-\infty}^{\infty} x_1[k] \sum_{\alpha=-\infty}^{\infty} x_2[\alpha] z^{-(\alpha+k)}$$

or

$$x[n] * x_2[n] \xrightarrow{z} \left( \sum_{k=-\infty}^{\infty} x_1[k] z^{-k} \right) \left( \sum_{\alpha=-\infty}^{\infty} x_2[\alpha] z^{-\alpha} \right)$$

$$x_1[n] * x_2[n] \xrightarrow{z} X_1(z) X_2(z)$$

6.5.9 Conjugation Property

If $x[n] \xrightarrow{z} X(z)$, with ROC : $R_x$
then $x^*[n] \xrightarrow{z} X^*(z^*)$, with ROC : $R_x$

If $x[n]$ is real, then

$$X(z) = X^*(z^*)$$
PROOF:
The z-transform of signal $x[n]$ is given by equation (6.1.1) as follows

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

(6.5.2)

Let z-transform of $x[n]$ is $X(z)$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

by taking complex conjugate on both sides of above equation

$$X^*(z) = \sum_{n=-\infty}^{\infty} [x[n]z^{-n}]^*$$

Replacing $z \rightarrow z^*$, we will get

$$X^*(z^*) = \sum_{n=-\infty}^{\infty} [x[n](z^*)^{-n}]^*$$

(6.5.3)

Comparing equation (6.5.2) and (6.5.3)

$$\mathcal{Z}\{x[n]\} = X^*(z^*)$$

(6.5.4)

For real $x[n]$, $x^*[n] = x[n]$, so

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z)$$

(6.5.5)

Comparing equation (6.5.4) and (6.5.5)

$$X(z) = X^*(z^*)$$

6.5.10 Initial Value Theorem

If $x[n] \xrightarrow{z} X(z)$, with ROC : $R_x$

then initial-value theorem states that

$$x[0] = \lim_{z \rightarrow \infty} X(z)$$

The initial-value theorem is valid only for the unilateral Laplace transform

PROOF:

For a causal signal $x[n]$

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

$$= x[0] + x[1]z^{-1} + x[2]z^{-2} + ...$$

Taking limit as $z \rightarrow \infty$ on both sides we get

$$\lim_{z \rightarrow \infty} X(z) = \lim_{z \rightarrow \infty} (x[0] + x[1]z^{-1} + x[2]z^{-2} + ...) = x[0]$$

$$x[0] = \lim_{z \rightarrow \infty} X(z)$$

6.5.11 Final Value Theorem

If $x[n] \xrightarrow{z} X(z)$, with ROC : $R_x$

then final-value theorem states that

$$x[\infty] = \lim_{z \rightarrow 1} (z - 1) X(z)$$

Final value theorem is applicable if $x(z)$ has no poles outside the unit circle. This theorem can be applied to either the unilateral or bilateral z-transform.
PROOF:

\[ \mathcal{Z}\{x[n+1]\} - \mathcal{Z}\{x[n]\} = \lim_{k \to \infty} \sum_{n=0}^{k} (x[n+1] - x[n]) z^{-n} \]

(6.5.6)

From the time shifting property of unilateral z-transform discussed in section 6.5.2

\[ x[n + n_0] \overset{z}{\longrightarrow} z^{n_0} \left( X(z) - \sum_{m=0}^{n_0-1} x[m] z^{-m} \right) \]

For \( n_0 = 1 \)

\[ x[n + 1] \overset{z}{\longrightarrow} z \left( X(z) - \sum_{m=0}^{0} x[m] z^{-m} \right) \]

\[ x[n + 1] \overset{z}{\longrightarrow} z(X(z) - x[0]) \]

Put above transformation in the equation (6.5.6)

\[ zX[z] - zx[0] - X[z] = \lim_{k \to \infty} \sum_{n=0}^{k} (x[n+1] - x[n]) z^{-n} \]

\[ (z - 1) X[z] - zx[0] = \lim_{k \to \infty} \sum_{n=0}^{k} (x[n+1] - x[n]) z^{-n} \]

Taking limit as \( z \to 1 \) on both sides we get

\[ \lim_{z \to 1} (z - 1) X[z] - x[0] = \lim_{k \to \infty} \sum_{n=0}^{k} (x[n+1] - x[n]) \]

\[ \lim_{z \to 1} (z - 1) X[z] - x[0] = \lim_{k \to \infty} \left\{ x[1] - x[0] + (x[2] - x[1]) + (x[3] - x[2]) + \ldots \right\} \]

\[ \lim_{z \to 1} (z - 1) X[z] - x[0] = x[\infty] - x[0] \]

Hence,

\[ X[\infty] = \lim_{z \to 1} (z - 1) X(z) \]

Summary of Properties

Let,

\[ x[n] \overset{z}{\longrightarrow} X(z), \quad \text{with} \quad \text{ROC} \quad R_x \]
\[ x_1[n] \overset{z}{\longrightarrow} X_1(z), \quad \text{with} \quad \text{ROC} \quad R_1 \]
\[ x_2[n] \overset{z}{\longrightarrow} X_2(z), \quad \text{with} \quad \text{ROC} \quad R_2 \]

The properties of z-transforms are summarized in the following table.

<table>
<thead>
<tr>
<th>Properties</th>
<th>Time domain</th>
<th>z-transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>( ax_1[n] + bx_2[n] )</td>
<td>( aX_1(z) + bX_2(z) )</td>
<td>at least ( R_1 \cap R_2 )</td>
</tr>
<tr>
<td>Time shifting (bilateral or non-causal)</td>
<td>( x[n - n_0] )</td>
<td>( z^{-n_0} X(z) )</td>
<td>( R_x ) except for the possible deletion or addition of ( z = 0 ) or ( z = \infty )</td>
</tr>
<tr>
<td></td>
<td>( x[n + n_0] )</td>
<td>( z^{n_0} X(z) )</td>
<td></td>
</tr>
<tr>
<td>Time shifting (unilateral or causal)</td>
<td>( x[n - n_0] )</td>
<td>( z^{-n_0} (X(z) + \sum_{m=1}^{n_0} x[-m] z^m) )</td>
<td>( R_x ) except for the possible deletion or addition of ( z = 0 ) or ( z = \infty )</td>
</tr>
<tr>
<td></td>
<td>( x[n + n_0] )</td>
<td>( z^{n_0} (X(z) - \sum_{m=0}^{n_0-1} x[m] z^{-m}) )</td>
<td></td>
</tr>
</tbody>
</table>
### 6.6 ANALYSIS OF DISCRETE LTI SYSTEMS USING Z-TRANSFORM

The z-transform is a very useful tool in the analysis of discrete LTI systems. As the Laplace transform is used in solving differential equations which describe continuous LTI systems, the z-transform is used to solve difference equations which describe the discrete LTI systems.

Similar to Laplace transform, for CT domain, the z-transform gives transfer function of the LTI discrete systems which is the ratio of the z-transform of the output variable to the z-transform of the input variable. These applications are discussed as follows.

#### 6.6.1 Response of LTI Continuous Time System

As discussed in chapter 4 (section 4.8), a discrete-time LTI system is always described by a linear constant coefficient difference equation given as follows:

\[
\sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{N} b_k x[n - k]
\]

where, \(a_k y[n - N] + a_{N-1} y[n - (N - 1)] + \ldots + a_1 y[n - 1] + a_0 y[n]\)

\[
= b_0 x[n - M] + b_{M-1} x[n - (M - 1)] + \ldots + b_1 x[n - 1] + b_0 x[n]
\]

(6.6.1)

where, \(N\) is order of the system.

The time-shift property of z-transform \(x[n - n_0] \rightarrow \underline{z}^{-n_0} X(z)\), is used to solve the above difference equation which converts it into an algebraic equation. By taking z-transform of above equation...

<table>
<thead>
<tr>
<th>Properties</th>
<th>Time domain</th>
<th>z-transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time reversal</td>
<td>(x[-n])</td>
<td>(X(1))</td>
<td>(1/R_x)</td>
</tr>
<tr>
<td>Differentiation in z domain</td>
<td>(nx[n])</td>
<td>(-z^dX(z)/dz)</td>
<td>(R_x)</td>
</tr>
<tr>
<td>Scaling in z domain</td>
<td>(a^n x[n])</td>
<td>(X(\frac{z}{a}))</td>
<td>(a</td>
</tr>
<tr>
<td>Time scaling (expansion)</td>
<td>(x_k[n] = x[n/k])</td>
<td>(X(z^k))</td>
<td>((R_x)^{1/k})</td>
</tr>
<tr>
<td>Time differencing</td>
<td>(x[n] - x[n-1])</td>
<td>((1 - z^{-1})X(z))</td>
<td>(R_x), except for the possible deletion of the origin</td>
</tr>
<tr>
<td>Time convolution</td>
<td>(x_1[n] * x_2[n])</td>
<td>(X_1(z)X_2(z))</td>
<td>at least (R_1 \cap R_2)</td>
</tr>
<tr>
<td>Conjugations</td>
<td>(x^*[n])</td>
<td>(X^<em>(z^</em>))</td>
<td>(R_x)</td>
</tr>
<tr>
<td>Initial-value theorem</td>
<td>(x[0] = \lim x(z))</td>
<td>provided if (n &lt; 0)</td>
<td></td>
</tr>
<tr>
<td>Final-value theorem</td>
<td>(x[\infty])</td>
<td>(= \lim x[n])</td>
<td>(= \lim (z-1)X(z))</td>
</tr>
</tbody>
</table>
The Z-Transform

### 6.6.1 Zero-input Response or Free Response or Natural Response

The zero-input response $y_{zi}[n]$ is mainly due to initial output in the system. The zero-input response is obtained from system equation (6.6.1) when input $x[n] = 0$.

By substituting $x[n] = 0$ and $y[n] = y_{zi}[n]$ in equation (6.6.1), we get

$$a_y y[n - N] + a_{N-1} y[n - (N - 1)] + \ldots + a_1 y[n - 1] + a_0 y[n] = 0$$

On taking $z$-transform of the above equation with given initial conditions, we can form an equation for $Y_{zi}(z)$. The zero-input response $y_{zi}[n]$ is given by inverse $z$-transform of $Y_{zi}(z)$.

**NOTE:**
The zero input response is also called the natural response of the system and it is denoted as $y_{N}[n]$. 

### 6.6.2 Zero-State Response or Forced Response

The zero-state response $y_{zs}[n]$ is the response of the system due to input signal and with zero initial conditions. The zero-state response is obtained from the difference equation (6.6.1) governing the system for specific input signal $x[n]$ for $n \geq 0$ and with zero initial conditions.

Substituting $y[n] = y_{zs}[n]$ in equation (6.6.1) we get,

$$a_y y_{zs}[n - N] + a_{N-1} y_{zs}[n - (N - 1)] + \ldots + a_1 y_{zs}[n - 1] + a_0 y_{zs}[n] = b_x x[n - M] + b_{M-1} x[n - (M - 1)] + \ldots + b_1 x[n - 1] + b_0 x[n]$$

Taking $z$-transform of the above equation with zero initial conditions for output (i.e., $y[-1] = y[-2] = 0$ we can form an equation for $Y_{zs}(z)$.

The zero-state response $y_{zs}[n]$ is given by inverse $z$-transform of $Y_{zs}(z)$.

**NOTE:**
The zero state response is also called the forced response of the system and it is denoted as $y_F[n]$. 

### Total Response

The total response $y[n]$ is the response of the system due to input signal and initial output. The total response can be obtained in following two ways:

Taking $z$-transform of equation (6.6.1) with non-zero initial conditions for both input and output, and then substituting for $X(z)$ we can form an equation for $Y(z)$. The total response $y[n]$ is given by inverse Laplace transform of $Y(s)$.

Alternatively, that total response $y[n]$ is given by sum of zero-input response $y_{zi}[n]$ and zero-state response $y_{zs}[n]$.

Therefore total response,

$$y[n] = y_{zi}[n] + y_{zs}[n]$$

### 6.6.2 Impulse Response and Transfer Function

System function or transfer function is defined as the ratio of the $z$-transform of the output $y[n]$ and the input $x[n]$ with zero initial conditions.

Let $x[n] \xrightarrow{Z} X(z)$ is the input and $y[n] \xrightarrow{Z} Y(z)$ is the output of an LTI discrete time system having impulse response $h(n) \xrightarrow{Z} H(z)$. The response
The Z-Transform

A linear time-invariant discrete time system is said to be causal if the impulse response \( h[n] = 0 \), for \( n < 0 \) and it is therefore right-sided. The ROC of such a system \( H(z) \) is the exterior of a circle. If \( H(z) \) is rational then the system is said to be causal if:

1. The ROC is the exterior of a circle outside the outermost pole; and
2. The degree of the numerator polynomial of \( H(z) \) should be less than or equal to the degree of the denominator polynomial.

**Stability**

A linear time-invariant discrete-time system is said to be BIBO stable if the impulse response \( h[n] \) is summable. That is

\[
\sum_{n=-\infty}^{\infty} |h[n]| < \infty
\]

The z-transform of \( h[n] \) is given as

\[
H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}
\]

Let \( z = e^{j\omega} \) (which describes a unit circle in the z-plane), then

\[
|H[e^{j\omega}]| = \left| \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \right| \\
\leq \sum_{n=-\infty}^{\infty} |h[n]e^{-j\omega n}| \\
= \sum_{n=-\infty}^{\infty} |h[n]| < \infty
\]

which is the condition for the stability. Thus we can conclude that
6.7.3 Stability and Causality

As we discussed previously, for a causal system with rational transfer function $H(z)$, the ROC is outside the outermost pole. For the BIBO stability the ROC should include the unit circle $|z| = 1$. Thus, for the system to be causal and stable these two conditions are satisfied if all the poles are within the unit circle in the $z$-plane.

6.8 BLOCK DIAGRAM REPRESENTATION

In $z$-domain, the input-output relation of an LTI discrete time system is represented by the transfer function $H(z)$, which is a rational function of $z$, as shown in equation

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \ldots + b_{M-1} z + b_M}{a_0 z^N + a_1 z^{N-1} + a_2 z^{N-2} + \ldots + a_{N-1} z + a_N}$$

where, $N = $ Order of the system, $M \leq N$, and $a_0 = 1$

The above transfer function is realized using unit delay elements, unit advance elements, adders and multipliers. Basic elements of block diagram with their $z$-domain representation is shown in table 6.3.

<table>
<thead>
<tr>
<th>Elements of Block diagram</th>
<th>Time Domain Representation</th>
<th>$s$-domain Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adder</td>
<td>$x_1[n] + x_2[n]$</td>
<td>$X_1(z) + X_2(z)$</td>
</tr>
<tr>
<td>Constant multiplier</td>
<td>$ax[n]$</td>
<td>$aX(z)$</td>
</tr>
<tr>
<td>Unit delay element</td>
<td>$z^{-1}x[n]$</td>
<td>$z^{-1}X(z)$</td>
</tr>
<tr>
<td>Unit advance element</td>
<td>$z x[n]$</td>
<td>$z X(z)$</td>
</tr>
</tbody>
</table>
The different types of structures for realizing discrete time systems are same as we discussed for the continuous-time system in the previous chapter.

### 6.8.1 Direct Form I Realization

Consider the difference equation governing the discrete time system with $a_0 = 1$,

$$y[n] + a_1 y[n - 1] + a_2 y[n - 2] + ... + a_N y[n - N] = b_0 x[n] + b_1 x[n - 1] + b_2 x[n - 2] + ... + b_M x[n - M]$$

Taking $Z$ transform of the above equation we get,

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - ... - a_N z^{-N} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + ... + b_M z^{-M} X(z) \quad (6.8.1)$$

The above equation of $Y(z)$ can be directly represented by a block diagram as shown in figure 6.8.1a. This structure is called direct form-I structure. This structure uses separate delay elements for both input and output of the system. So, this realization uses more memory.

![General structure of direct form-realization](image.png)

**For example consider a discrete LTI system which has the following impulse response**

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + 2z^{-2}}{1 + 4z^{-1} + 3z^{-2}}$$

$$Y(z) + 4z^{-1} Y(z) + 3z^{-2} Y(z) = 1X(z) + 2z^{-1} X(z) + 2z^{-2} X(z)$$

Comparing with standard form of equation (6.8.1), we get $a_1 = 4$, $a_2 = 3$ and $b_0 = 1$, $b_1 = 2$, $b_2 = 2$. Now put these values in general structure of Direct form-I realization we get
6.8.2 Direct Form II Realization

Consider the general difference equation governing a discrete LTI system
\[ y[n] + a_1 y[n-1] + a_2 y[n-2] + \ldots + a_n y[n-N] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + \ldots + b_M x[n-M] \]
Taking \( Z \)-transform of the above equation we get,
\[ Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \ldots - a_n z^{-N} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \ldots + b_M z^{-M} X(z) \]
It can be simplified as,
\[ Y(z)[1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{-N}] = X(z)[b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}] \]
Let,
\[ Y(z) = W(z) \times \frac{1}{X(z)} \]
where,
\[ W(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{-N}} \quad \text{(6.8.2)} \]
\[ Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{X(z)} \quad \text{(6.8.3)} \]
Equation (6.8.2) can be simplified as,
\[ W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) + \ldots + a_n z^{-N} W(z) = X(z) \]
\[ W(z) = \frac{X(z)}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{-N}} \quad \text{(6.8.4)} \]
Similarly by simplifying equation (6.8.3), we get
\[ Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + \ldots + b_M z^{-M} W(z) \]
\[ Y(z) = W(z) \quad \text{(6.8.5)} \]
Equation (6.8.4) and (6.8.5) can be realized together by a direct structure called direct form-II structure as shown in figure 6.8.2a. It uses less number of delay elements than the Direct Form I structure.

For example, consider the same transfer function \( H(z) \) which is discussed above
\[ H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + 2z^{-2}}{1 + 4z^{-1} + 3z^{-2}} \]
Let
\[ Y(z) = W(z) \times \frac{1}{X(z)} \]
where,
\[ W(z) = \frac{1}{1 + 4z^{-1} + 3z^{-2}}, \quad Y(z) = \frac{1}{W(z)} \]
so,
\[ W(z) = X(z) - 4z^{-1} W(z) - 3z^{-2} W(z) \]
and
\[ Y(z) = W(z) \]
Comparing these equations with standard form of equation (6.8.4) and (6.8.5), we have \( a_1 = 4, a_2 = 3 \) and \( b_0 = 1, b_1 = 2, b_2 = 2 \). Substitute these
values in general structure of Direct form II, we get as shown in figure 6.8.2b.

Figure 6.8.2a General structure of direct form-II realization

Figure 6.8.2b

6.8.3 Cascade Form

The transfer function \( H(z) \) of a discrete time system can be expressed as a product of several transfer functions. Each of these transfer functions is realized in direct form-I or direct form II realization and then they are cascaded.

Consider a system with transfer function

\[
H(z) = \frac{(b_0 + b_1z^{-1} + b_2z^{-2})(b_{m0} + b_{m1}z^{-1} + b_{m2}z^{-2})}{(1 + a_{11}z^{-1} + a_{12}z^{-2})(1 + a_{m1}z^{-1} + a_{m2}z^{-2})} = H_1(z)H_2(z)
\]

where

\[
H_1(z) = \frac{b_0 + b_1z^{-1} + b_2z^{-2}}{1 + a_{11}z^{-1} + a_{12}z^{-2}}
\]

\[
H_2(z) = \frac{b_{m0} + b_{m1}z^{-1} + b_{m2}z^{-2}}{1 + a_{m1}z^{-1} + a_{m2}z^{-2}}
\]
The Z-Transform

Realizing $H_1(z)$ and $H_2(z)$ in direct form II and cascading we obtain cascade form of the system function $H(z)$ as shown in figure 6.8.3.

**6.8.4 Parallel Form**

The transfer function $H(z)$ of a discrete time system can be expressed as the sum of several transfer functions using partial fractions. Then the individual transfer functions are realized in direct form I or direct form II realization and connected in parallel for the realization of $H(z)$. Let us consider the transfer function

$$H(z) = c + \frac{c_1}{1 - p_1z^{-1}} + \frac{c_2}{1 - p_2z^{-1}} + \ldots + \frac{c_N}{1 - p_Nz^{-1}}$$

Now each factor in the system is realized in direct form II and connected in parallel as shown in figure 6.8.4.

**6.9 RELATIONSHIP BETWEEN S-PLANE & -PLANE**

There exists a close relationship between the Laplace and $z$-transforms. We
know that a DT sequence $x[n]$ is obtained by sampling a CT signal $x(t)$ with a sampling interval $T$, the CT sampled signal $x_s(t)$ is written as follows

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT)$$

where $x(nT)$ are sampled values of $x(t)$ which equals the DT sequence $x[n]$. Taking the Laplace transform of $x_s(t)$, we have

$$X(s) = \mathcal{L}\{x_s(t)\} = \sum_{n=-\infty}^{\infty} x(nT) \mathcal{L}\{\delta(t - nT)\}$$

$$= \sum_{n=-\infty}^{\infty} X(nT) e^{-nTs} \quad (6.9.1)$$

The $z$-transform of $x[n]$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad (6.9.2)$$

Comparing equation (6.9.1) and (6.9.2)

$$X(s) = X(z) \bigg|_{e^{-sT}} \quad x[n] = x(nT)$$

***********
EXERCISE 6.1

MCQ 6.1.1 The z-transform and its ROC of a discrete time sequence

\[ x[n] = \begin{cases} \left(-\frac{1}{z}\right)^n, & n < 0 \\ 0, & n \geq 0 \end{cases} \]

will be

(A) \( \frac{2z}{z - 1} \), \( |z| > \frac{1}{2} \)

(B) \( \frac{z}{z - 1} \), \( |z| < \frac{1}{2} \)

(C) \( \frac{2z}{z - 1} \), \( |z| < \frac{1}{2} \)

(D) \( \frac{2z}{z - 1} \), \( |z| > \frac{1}{2} \)

MCQ 6.1.2 The ROC of z-transform of the discrete time sequence \( x[n] = \left(\frac{1}{4}\right)^n \) is

(A) \( \frac{1}{2} < |z| < 2 \)

(B) \( |z| > 2 \)

(C) \( -2 < |z| < 2 \)

(D) \( |z| < \frac{1}{2} \)

MCQ 6.1.3 Consider a discrete-time signal \( x[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[-n - 1] \). The ROC of its z-transform is

(A) \( |z| > \frac{1}{3} \)

(B) \( \frac{1}{3} < |z| < \frac{1}{2} \)

(C) \( |z| > \frac{1}{2} \)

(D) \( \frac{1}{3} < |z| < \frac{1}{2} \)

MCQ 6.1.4 For a signal \( x[n] = [\alpha^n + \alpha^{-n}]u[n] \), the ROC of its z-transform would be

(A) \( |z| > \min\left(\frac{1}{|\alpha|}, \frac{1}{|-\alpha|}\right) \)

(B) \( |z| > |\alpha| \)

(C) \( |z| < \max\left(\frac{1}{|\alpha|}, \frac{1}{1 - |\alpha|}\right) \)

(D) \( |z| < |\alpha| \)

MCQ 6.1.5 Match List I (discrete time sequence) with List II (z-transform) and choose the correct answer using the codes given below the lists:

**List-I (Discrete Time Sequence)**

P. \( u[n - 2] \)

Q. \( -u[-n - 3] \)

R. \( u[n + 4] \)

S. \( u[-n] \)

**List-II (z-Transform)**

1. \( \frac{1}{z^{-2}(1 - z^{-1})} \), \( |z| < 1 \)

2. \( \frac{-z^{-1}}{1 - z^{-1}} \), \( |z| < 1 \)

3. \( \frac{1}{z^{-4}(1 - z^{-1})} \), \( |z| > 1 \)

4. \( \frac{z^2}{1 - z^{-1}} \), \( |z| > 1 \)

**Codes:**

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>(B)</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>(C)</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>(D)</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>
MCQ 6.1.6
The z-transform of signal \( x[n] = e^{jn}u[n] \) is
(A) \( \frac{z}{z + 1} \), ROC: \( |z| > 1 \)
(B) \( \frac{z}{z - 1} \), ROC: \( |z| > 1 \)
(C) \( \frac{z}{z + 1} \), ROC: \( |z| < 1 \)
(D) \( \frac{1}{z + 1} \), ROC: \( |z| < 1 \)

MCQ 6.1.7
Consider the pole zero diagram of an LTI system shown in the figure which corresponds to transfer function \( H(z) \).

Match List I (The impulse response) with List II (ROC which corresponds to above diagram) and choose the correct answer using the codes given below:
{Given that \( H(1) = 1 \)}

<table>
<thead>
<tr>
<th>List-I (Impulse Response)</th>
<th>List-II (ROC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P. ( {[-4]2^n + 6(-3)^n}u[n] )</td>
<td>1. does not exist</td>
</tr>
<tr>
<td>Q. ( {[-4]2^n + (-6)3^n}u[-n - 1] )</td>
<td>2. (</td>
</tr>
<tr>
<td>R. ( {4(2^n)u[-n - 1] + (-6)3^n}u[n] )</td>
<td>3. (</td>
</tr>
<tr>
<td>S. ( {4(2^n)u[-n - 1] + (-6)3^n}u[-n - 1] )</td>
<td>4. (2 &lt;</td>
</tr>
</tbody>
</table>

Codes:
(A) 4 1 3 2
(B) 2 1 3 4
(C) 1 4 2 3
(D) 2 4 3 1

MCQ 6.1.8
The z-transform of a signal \( x[n] \) is \( X(z) = e^{jz} + e^{jz} \), \( |z| \neq 0 \). \( x[n] \) would be
(A) \( \delta[n] + \frac{1}{n!} \)
(B) \( u[n] + \frac{1}{n!} \)
(C) \( u[n - 1] + n! \)
(D) \( \delta[n] + (n - 1)! \)

Common Data For Q. 9 to 11:
Consider a discrete time signal \( x[n] \) and its z-transform \( X(z) \) given as
\[ X(z) = \frac{z^2 + 5z}{z^2 - 2z - 3} \]

MCQ 6.1.9
If ROC of \( X(z) \) is \(|z| < 1 \), then signal \( x[n] \) would be
(A) \( \{-2(3)^n + (-1)^n\}u[-n - 1] \)
(B) \( \{2(3)^n - (-1)^n\}u[n] \)
(C) \( \{-2(3)^n + (-1)^n\}u[n] \)
(D) \( \{2(3)^n + 1\}u[n] \)

MCQ 6.1.10
If ROC of \( X(z) \) is \(|z| > 3 \), then signal \( x[n] \) would be
(A) \( \{2(3)^n - (-1)^n\}u[n] \)
(B) \( \{-2(3)^n + (-1)^n\}u[-n - 1] \)
(C) \( \{-2(3)^n + (-1)^n\}u[n] \)
(D) \( \{2(3)^n + 1\}u[n] \)
MCQ 6.1.11 If ROC of \( X(z) \) is \( 1 < |z| < 3 \), the signal \( x[n] \) would be

(A) \( 2(3)^n - (-1)^n u[n] \)

(B) \( -2(3)^n + (-1)^n u[-n-1] \)

(C) \( -2(3)^n u[-n-1] - (-1)^n u[n] \)

(D) \( 2(3)^n + (-1)^n u[-n-1] \)

MCQ 6.1.12 Consider a DT sequence \( x[n] = x_1[n] + x_2[n] \) where, \( x_1[n] = (0.7)^n u[n-1] \) and \( x_2[n] = (-0.4)^n u[n-2] \). The region of convergence of \( z \)-transform of \( x[n] \) is

(A) \( |z| > 0.4 \)

(B) \(|z| < 0.4 \)

(C) none of these

MCQ 6.1.13 The \( z \)-transform of a DT signal \( x[n] \) is

\( X(z) = \frac{z}{8z^2 - 2z - 1} \). What will be the \( z \)-transform of \( x[n-4] \)?

(A) \( \frac{z+4}{8(z+4)^2 - 2(z+4) - 1} \)

(B) \( \frac{z^2}{8z^2 - 2z - 1} \)

(C) \( \frac{4z}{128z^2 - 8z - 1} \)

(D) \( \frac{1}{8z^2 - 2z^2 + z^2} \)

MCQ 6.1.14 Let \( x_1[n] \), \( x_2[n] \) and \( x_3[n] \) be three discrete time signals and \( X_1(z) \), \( X_2(z) \) and \( X_3(z) \) are their \( z \)-transform respectively given as

\[
X_1(z) = \frac{z^2}{(z-1)(z-0.5)},
\]

\[
X_2(z) = \frac{z}{(z-1)(z-0.5)},
\]

and

\[
X_3(z) = \frac{1}{(z-1)(z-0.5)}.
\]

Then \( x_1[n] \), \( x_2[n] \) and \( x_3[n] \) are related as

(A) \( x_1[n-2] = x_2[n-1] - x_3[n] \)

(B) \( x_1[n+2] = x_2[n+1] = x_3[n] \)

(C) \( x_1[n] = x_2[n-1] = x_3[n-2] \)

(D) \( x_1[n+1] = x_2[n-1] = x_3[n] \)

MCQ 6.1.15 The \( z \)-transform of the discrete time signal \( x[n] \) shown in the figure is

\[
X(z) = \frac{z^{-k}}{1 - z^{-1}}.
\]

(A) \( \frac{z^{-k}}{1 - z^{-1}} \)

(B) \( \frac{1}{1 + z^{-1}} \)

(C) \( \frac{1 - z^{-k}}{1 - z^{-1}} \)

(D) \( \frac{1 + z^{-k}}{1 - z^{-1}} \)

MCQ 6.1.16 Consider the unilateral \( z \)-transform pair \( x[n] \leftrightarrow X(z) = \frac{z}{z-1} \). The \( z \)-transform of \( x[n-1] \) and \( x[n+1] \) are respectively

(A) \( \frac{z^2}{z-1}, \frac{1}{z-1} \)

(B) \( \frac{1}{z-1}, \frac{z^2}{z-1} \)

(C) \( \frac{1}{z-1}, \frac{z}{z-1} \)

(D) \( \frac{z}{z-1}, \frac{z^2}{z-1} \)

MCQ 6.1.17 A discrete time causal signal \( x[n] \) has the \( z \)-transform

\[ X(z) = \frac{z}{z-0.4}, \text{ ROC: } |z| > 0.4 \]

The ROC for \( z \)-transform of the even part of \( x[n] \) will be
MCQ 6.1.18
Match List I (Discrete time sequence) with List II (z-transform) and select the correct answer using the codes given below the lists.

<table>
<thead>
<tr>
<th>List-I (Discrete time sequence)</th>
<th>List-II (z-transform)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P. ((-1)^nu[n])</td>
<td>1. (\frac{z^{-1}}{(1-z^{-1})^2}), ROC: (</td>
</tr>
<tr>
<td>Q. (-nu[- n - 1])</td>
<td>2. (\frac{1}{1+z^{-1}}), ROC: (</td>
</tr>
<tr>
<td>R. ((-1)^nu[n])</td>
<td>3. (\frac{z^{-1}}{(1-z^{-1})^2}), ROC: (</td>
</tr>
<tr>
<td>S. (nu[n])</td>
<td>4. (-\frac{z^{-1}}{(1+z^{-1})^2}), ROC: (</td>
</tr>
</tbody>
</table>

Codes:
(A) 4 1 2 3
(B) 4 3 2 1
(C) 3 1 4 2
(D) 2 4 1 3

MCQ 6.1.19
A discrete time sequence is defined as \(x[n] = \frac{1}{3}(-2)^{-n}u[-n - 1]\). The z-transform of \(x[n]\) is
(A) \(\log(z + \frac{1}{2})\), ROC: \(|z| < \frac{1}{2}\)
(B) \(\log(z - \frac{1}{2})\), ROC: \(|z| > \frac{1}{2}\)
(C) \(\log(z - 2)\), ROC: \(|z| > 2\)
(D) \(\log(z + 2)\), ROC: \(|z| < 2\)

MCQ 6.1.20
Consider a z-transform pair \(x[n] \rightarrow X(z)\) with ROC \(R_x\). The z transform and its ROC for \(y[n] = a^n x[n]\) will be
(A) \(X\left(\frac{z}{a}\right)\), ROC: \(|a| R_x\)
(B) \(X(z + a)\), ROC: \(R_x\)
(C) \(z^{-a}X(z)\), ROC: \(R_x\)
(D) \(X(az)\), ROC: \(|a| R_x\)

MCQ 6.1.21
Let \(X(z)\) be the z-transform of a causal signal \(x[n] = a^n u[n]\) with ROC: \(|z| > a\). Match the discrete sequences \(S_1, S_2, S_3\) and \(S_4\) with ROC of their z-transforms \(R_1, R_2\) and \(R_3\).

<table>
<thead>
<tr>
<th>Sequences</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_1): (x[n-2])</td>
<td>(R_1): (</td>
</tr>
<tr>
<td>(S_2): (x[n+2])</td>
<td>(R_2): (</td>
</tr>
<tr>
<td>(S_3): (x[-n])</td>
<td>(R_3): (</td>
</tr>
<tr>
<td>(S_4): ((-1)^n x[n])</td>
<td>(A) ((S_2, R_2), (S_3, R_3), (S_4, R_3))</td>
</tr>
<tr>
<td></td>
<td>(B) ((S_2, R_3), (S_3, R_3), (S_4, R_1))</td>
</tr>
<tr>
<td></td>
<td>(C) ((S_2, R_3), (S_3, R_3), (S_4, R_3))</td>
</tr>
<tr>
<td></td>
<td>(D) ((S_2, R_3), (S_3, R_3), (S_4, R_3))</td>
</tr>
</tbody>
</table>
Consider a discrete time signal \( x[n] = \alpha^n u[n] \) and its z-transform \( X(z) \). Match List I (discrete signals) with List II (z-transform) and select the correct answer using the codes given below:

**List-I (Discrete time signal)**

**List-II (z-transform)**

- **P.** \( x[n/2] \)
- **Q.** \( x[n - 2]u[n - 2] \)
- **R.** \( x[n + 2]u[n] \)
- **S.** \( \beta^{2n}x[n] \)

**Codes:**

(A) 1 2 4 3  
(B) 2 4 1 3  
(C) 1 4 2 3  
(D) 2 1 4 3

**MCQ 6.1.23**

The z-transform of a discrete sequence \( x[n] \) is \( X(z) \), then the z-transform of \( x[2n] \) will be

(A) \( X(2z) \)  
(B) \( \frac{X(z)}{2} \)  
(C) \( \frac{X(\sqrt{z}) + X(-\sqrt{z})}{2} \)  
(D) \( X(\sqrt{z}) \)

**MCQ 6.1.24**

Consider a signal \( x[n] \) and its z transform \( X(z) \) given as \( X(z) = \frac{4z}{8z^2 - 2z - 1} \). The z-transform of the sequence \( y[n] = x[0] + x[1] + x[2] + \ldots + x[n] \) will be

(A) \( \frac{4z^2}{(z-1)(8z^2 - 2z - 1)} \)  
(B) \( \frac{4z(z-1)}{8z^2 - 2z - 1} \)  
(C) \( \frac{4z^2}{(z+1)(8z^2 - 2z - 1)} \)  
(D) \( \frac{4z(z+1)}{8z^2 - 2z - 1} \)

**MCQ 6.1.25**

What is the convolution of two DT sequence \( x[n] = \{-1, 2, 0, 3\} \) and \( h[n] = \{2, 0, 3\} \)?

(A) \( \{-2, -4, 3, 6, 9\} \)  
(B) \( \{-2, 4, -3, 12, 0, 9\} \)  
(C) \( \{9, 6, 3, -4, -2\} \)  
(D) \( \{-3, 6, 7, 4, 6\} \)

**MCQ 6.1.26**

If \( x[n] \to X(z) \) be a z-transform pair, then which of the following is true?

(A) \( x[n] \to X^*(-z) \)  
(B) \( x[n] \to X^*(-z^*) \)  
(C) \( x[n] \to X^*(z^*) \)  
(D) \( x[n] \to X^*(-z^*) \)

**MCQ 6.1.27**

A discrete-time system with input \( x[n] \) and output \( y[n] \) is governed by following difference equation

\[ y[n] - \frac{1}{2}y[n-1] = x[n], \text{ with initial condition } y[-1] = 3 \]

The impulse response of the system

(A) \( \frac{5}{2(2^n - 1)}, \ n \geq 0 \)  
(B) \( \frac{5}{2(2^n)} \), \( n \geq 0 \)  
(C) \( \frac{5}{2(2^n - 1)}, \ n \geq 0 \)  
(D) \( \frac{5}{2(2^n+1)} \), \( n \geq 0 \)
MCQ 6.1.28
Consider a causal system with impulse response \( h[n] = (2)^n u[n] \). If \( x[n] \) is the input and \( y[n] \) is the output to this system, then which of the following difference equation describes the system?

(A) \( y[n] + 2y[n+1] = x[n] \)
(B) \( y[n] - 2y[n-1] = x[n] \)
(C) \( y[n] + 2y[n-1] = x[n] \)
(D) \( y[n] - \frac{1}{2}y[n-1] = x[n] \)

MCQ 6.1.29
The impulse response of a system is given as \( h[n] = \delta[n] - (\frac{1}{2})^n u[n] \). For an input \( x[n] \) and output \( y[n] \), the difference equation that describes the system is

(A) \( y[n] + 2y[n-1] = 2x[n] \)
(B) \( y[n] + 0.5y[n-1] = 0.5x[n-1] \)
(C) \( y[n] + 2ny[n-1] = x[n] \)
(D) \( y[n] - 0.5y[n-1] = 0.5x[n-1] \)

MCQ 6.1.30
The input-output relationship of a system is given as \( y[n] = 0.4y[n-1] = x[n] \) where, \( x[n] \) and \( y[n] \) are the input and output respectively. The zero state response of the system for an input \( x[n] = (0.4)^n u[n] \) is

(A) \( n(0.4)^n u[n] \)
(B) \( n^2(0.4)^n u[n] \)
(C) \( (n + 1)(0.4)^n u[n] \)
(D) \( \frac{1}{n}(0.4)^n u[n] \)

MCQ 6.1.31
A discrete time system has the following input-output relationship

\[ y[n] = \frac{1}{2} - (2)^n \] \( u[n] \)

If an input \( x[n] = u[n] \) is applied to the system, then its zero state response will be

(A) \( 2 - \left(\frac{1}{2}\right)^n \) \( u[n] \)
(B) \( 2 - \left(\frac{1}{2}\right)^n \) \( u[n] \)
(C) \( \frac{1}{2} - \left(\frac{1}{2}\right)^n \) \( u[n] \)
(D) \( 2 - (2)^n u[n] \)

MCQ 6.1.32
Consider the transfer function of a system

\[ H(z) = \frac{2z(z-1)}{z^2 + 4z + 4} \]

For an input \( x[n] = 2\delta[n] + \delta[n+1] \), the system output is

(A) \( 2\delta[n+1] + 6(2)^n u[n] \)
(B) \( 2\delta[n] - 6(-2)^n u[n] \)
(C) \( 2\delta[n+1] - 6(-2)^n u[n] \)
(D) \( 2\delta[n+1] + 6\left(\frac{1}{2}\right)^n u[n] \)

MCQ 6.1.33
The transfer function of a discrete time LTI system is given as

\[ H(z) = \frac{z}{z^2 + 1}, \quad \text{ROC : } |z| > 1 \]

Consider the following statements:
1. The system is causal and BIBO stable.
2. The system is causal but BIBO unstable.
3. The system is non-causal and BIBO unstable.
4. Impulse response \( h[n] = \sin\left(\frac{\pi}{2}n\right) u[n] \)

Which of the above statements are true?
(A) 1 and 4
(B) 2 and 4
(C) 1 only
(D) 3 and 4

MCQ 6.1.34
Which of the following statement is not true?
A LTI system with rational transfer function \( H(z) \) is
(A) causal if the ROC is the exterior of a circle outside the outermost pole.
(B) stable if the ROC of \( H(z) \) includes the unit circle \( |z| = 1 \).
(C) causal and stable if all the poles of \( H(z) \) lie inside unit circle.
(D) none of above
MCQ 6.1.35
If \( h[n] \) denotes the impulse response of a causal system, then which of the following system is not stable?

(A) \( h[n] = n \left( \frac{1}{2^n} \right) u[n] \)

(B) \( h[n] = \left( \frac{1}{2^n} \right) u[n] \)

(C) \( h[n] = \delta[n] - \left( \frac{1}{3} \right)^n u[n] \)

(D) \( h[n] = (2^n - (3)^n)u[n] \)

MCQ 6.1.36
A causal system with input \( x[n] \) and output \( y[n] \) has the following relationship

\[ y[n] + 3y[n - 1] + 2y[n - 2] = 2x[n] + 3x[n - 1] \]

The system is

(A) stable

(B) unstable

(C) marginally stable

(D) none of these

MCQ 6.1.37
A causal LTI system is described by the difference equation

\[ y[n] = x[n] + y[n - 1] \]

Consider the following statement

1. Impulse response of the system is \( h[n] = u[n] \)

2. The system is BIBO stable

3. For an input \( x[n] = (0.5)^n u[n] \), system output is \( y[n] = 2u[n] - (0.5)^n u[n] \)

Which of the above statements is/are true?

(A) 1 and 2

(B) 1 and 3

(C) 2 and 3

(D) 1, 2 and 3

MCQ 6.1.38
Match List I (system transfer function) with List II (property of system) and choose the correct answer using the codes given below

<table>
<thead>
<tr>
<th>List-I (System transfer function)</th>
<th>List-II (Property of system)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P. ( H(z) = \frac{z^2}{(z - 1.2)^2} ), ( ROC :</td>
<td>z</td>
</tr>
<tr>
<td>Q. ( H(z) = \frac{z^2}{(z - 1.2)^2} ), ( ROC :</td>
<td>z</td>
</tr>
<tr>
<td>R. ( H(z) = \frac{z^4}{(z - 0.8)^2} ), ( ROC :</td>
<td>z</td>
</tr>
<tr>
<td>S. ( H(z) = \frac{z^2}{(z - 0.8)^2} ), ( ROC :</td>
<td>z</td>
</tr>
</tbody>
</table>

Codes:

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>(B)</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>(C)</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>(D)</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

MCQ 6.1.39
The transfer function of a DT feedback system is

\[ H(z) = \frac{P}{1 + P \left( \frac{z}{z - 0.9} \right)} \]

The range of \( P \), for which the system is stable will be

(A) \(-1.9 < P < -0.1\)

(B) \( P < 0 \)

(C) \( P > -1 \)

(D) \( P > -0.1 \) or \( P < -1.9 \)

*Shipping Free*
Consider three stable LTI systems \( S_1, S_2 \) and \( S_3 \) whose transfer functions are

\[
S_1 : H(z) = \frac{z - \frac{1}{z}}{2z^2 + \frac{1}{z} - \frac{3}{36}} \\
S_2 : H(z) = \frac{z + 1}{-\frac{3}{2}z^2 - \frac{3}{2}z^2 + \frac{3}{3} + z} \\
S_3 : H(z) = \frac{1 + \frac{1}{2}z^{-2} - \frac{1}{2}z^{-1}}{z^2(1 - \frac{1}{3}z)(1 - \frac{1}{3}z)}
\]

Which of the above systems is/are causal?

(A) \( S_1 \) only  
(B) \( S_1 \) and \( S_2 \)  
(C) \( S_1 \) and \( S_3 \)  
(D) \( S_1, S_2 \) and \( S_3 \)

The \( z \)-transform of \( \delta[n - k], k > 0 \) is

(A) \( z^k, z > 0 \)  
(B) \( z^k, z > 0 \)  
(C) \( z^k, z \neq 0 \)  
(D) \( z^k, z \neq 0 \)

The \( z \)-transform of \( \delta[n + k], k > 0 \) is

(A) \( z^{-k}, z \neq 0 \)  
(B) \( z^{-k}, z \neq 0 \)  
(C) \( z^{-k}, \text{ all } z \)  
(D) \( z^{-k}, \text{ all } z \)

The \( z \)-transform of \( u[n] \) is

(A) \( \frac{1}{1 - z^{-1}}, \text{ } \quad |z| > 1 \)  
(B) \( \frac{-1}{1 - z^{-1}}, \text{ } \quad |z| < 1 \)  
(C) \( \frac{z}{1 - z^{-1}}, \text{ } \quad |z| < 1 \)  
(D) \( \frac{z}{1 - z^{-1}}, \text{ } \quad |z| > 1 \)

The \( z \)-transform of \( \left( \frac{1}{4} \right)^n (u[n] - u[n-5]) \) is

(A) \( \frac{z^5 - (0.25)^5}{z^5(z - 0.25)^5}, \text{ } |z| > 0.25 \)  
(B) \( \frac{z^5 - (0.25)^5}{z^5(z - 0.25)^5}, \text{ } |z| > 0.5 \)  
(C) \( \frac{z^5 - (0.25)^5}{z^5(z - 0.25)^5}, \text{ } |z| < 0.25 \)  
(D) \( \frac{z^5 - (0.25)^5}{z^5(z - 0.25)^5}, \text{ } \text{all } z \)

The \( z \)-transform of \( \left( \frac{1}{4} \right)^n u[-n] \) is

(A) \( \frac{4^z}{4^z - 1}, \text{ } |z| > \frac{1}{4} \)  
(B) \( \frac{4^z}{4^z - 1}, \text{ } |z| < \frac{1}{4} \)  
(C) \( \frac{1}{1 - 4^z}, \text{ } |z| > \frac{1}{4} \)  
(D) \( \frac{1}{1 - 4^z}, \text{ } |z| < \frac{1}{4} \)

The \( z \)-transform of \( 3^n u[-n - 1] \) is

(A) \( \frac{z}{3 - z}, \text{ } |z| > 3 \)  
(B) \( \frac{z}{3 - z}, \text{ } |z| < 3 \)  
(C) \( \frac{3}{3 - z}, \text{ } |z| > 3 \)  
(D) \( \frac{3}{3 - z}, \text{ } |z| < 3 \)

The \( z \)-transform of \( \left( \frac{2}{3} \right)^n \) is

(A) \( \frac{-5z}{(2z - 3)(3z - 2)}, \text{ } \quad \frac{3}{2} < |z| < -\frac{2}{3} \)  
(B) \( \frac{-5z}{(2z - 3)(3z - 2)}, \text{ } \quad \frac{3}{2} < |z| < \frac{2}{3} \)  
(C) \( \frac{5z}{(2z - 3)(3z - 2)}, \text{ } \quad \frac{3}{2} < |z| < -\frac{2}{3} \)  
(D) \( \frac{5z}{(2z - 3)(3z - 2)}, \text{ } \quad \frac{3}{2} < |z| < -\frac{2}{3} \)
MCQ 6.1.48 The $z$-transform of $\cos\left(\frac{\pi}{3}n\right)u[n]$ is

(A) $\frac{z(2z-1)}{2(z^2-z+1)}$, $0 < |z| < 1$

(B) $\frac{z(2z-1)}{2(z^2-z+1)}$, $|z| > 1$

(C) $\frac{z(1-2z)}{2(z^2-z+1)}$, $0 < |z| < 1$

(D) $\frac{z(1-2z)}{2(z^2-z+1)}$, $|z| > 1$

MCQ 6.1.49 The $z$-transform of $\{3, 0, 0, 0, 0, 0, 1, 4\}$ is

(A) $3z^6 + 4z^5 + 7z^2 + z^3$

(B) $3z^5 + 6z^4 - 4z^2$

(C) $3z^5 + 6 + z - 4z^2$

(D) $3z^5 + 6 + z - 4z^2$

MCQ 6.1.50 The $z$-transform of $x[n] = \{2, 4, 5, 7, 0, 1\}$ is

(A) $2z^2 + 4z + 5 + 7z + z^3$

(B) $2z^2 + 4z + 3 + 5z + z^3$

(C) $2z^2 + 4z^3 + 5 + 7z + z^3$

(D) $2z^2 + 4z^3 + 5 + 7z + z^3$

MCQ 6.1.51 The $z$-transform of $x[n] = \{1, 0, -1, 0, 1, -1\}$ is

(A) $1 + 2z^{-2} - 4z^{-4} + 5z^{-5}$, $z \neq 0$

(B) $1 - z^{-2} + z^{-4} - z^{-5}$, $z \neq 0$

(C) $1 - 2z^2 + 4z^4 - 5z^6$, $z \neq 0$

(D) $1 - z^2 + z^4 - z^6$, $z \neq 0$

MCQ 6.1.52 The time signal corresponding to $\frac{z^2 - 3z}{z^2 + \frac{3}{2}z - 1}$, $\frac{1}{2} < |z| < 2$ is

(A) $\frac{1}{z}u[n] - 2^{-n+1}u[-n-1]$

(B) $\frac{1}{z}u[n] - 2^{-n+1}u[n+1]$

(C) $\frac{1}{z}u[n] + 2^{-n+1}u[n+1]$

(D) $\frac{1}{z}u[n] - 2^{-n-1}u[-n-1]$

MCQ 6.1.53 The time signal corresponding to $\frac{3z^2 - \frac{1}{2}z}{z^2 - 16}$, $|z| > 4$ is

(A) $\left[\frac{49}{32}(-4)^n + \frac{47}{32}4^n\right]u[n]$

(B) $\left[\frac{49}{32}4^n + \frac{47}{32}4^n\right]u[n]$

(C) $\left[\frac{49}{32}(-4)^n u[-n] + \frac{47}{32}4^n u[n]\right]$

(D) $\left[\frac{49}{32}4^n u[n] + \frac{47}{32}(-4)^n u[-n]\right]$

MCQ 6.1.54 The time signal corresponding to $\frac{2z^4 - 2z^2 - 2z^2}{z^2 - 1}$, $|z| > 1$ is

(A) $2\delta[n - 2] + [1 - (-1)^n]u[n - 2]$

(B) $2\delta[n + 2] + [1 - (-1)^n]u[n + 2]$

(C) $2\delta[n + 2] + [(-1)^n - 1]u[n + 2]$

(D) $2\delta[n - 2] + [(-1)^n - 1]u[n - 2]$

MCQ 6.1.55 The time signal corresponding to $1 + 2z^{-6} + 4z^{-8}$, $|z| > 0$ is

(A) $\delta[n] + 2\delta[n - 6] + 4\delta[n - 8]$

(B) $\delta[n] + 2\delta[n + 6] + 4\delta[n + 8]$

(C) $\delta[-n] + 2\delta[-n - 6] + 4\delta[-n - 8]$

(D) $\delta[-n] + 2\delta[-n - 6] + 4\delta[-n - 8]$
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MCQ 6.1.56 The time signal corresponding to \[ \sum_{k=5}^{10} \frac{1}{k} z^k \], \(|z| > 0\) is

(A) \[ \sum_{k=5}^{10} \frac{1}{k} \delta[n + k] \]  
(B) \[ \sum_{k=5}^{10} \frac{1}{k} \delta[n - k] \]  
(C) \[ \sum_{k=5}^{10} \frac{1}{k} \delta[-n + k] \]  
(D) \[ \sum_{k=5}^{10} \frac{1}{k} \delta[-n - k] \]

MCQ 6.1.57 The time signal corresponding to \((1 + z^{-3})^3\), \(|z| > 0\) is

(A) \[ \delta[-n] + 3\delta[-n - 1] + 3\delta[-n - 2] + \delta[-n - 3] \]  
(B) \[ \delta[-n] + 3\delta[-n + 1] + 3\delta[-n + 2] + \delta[-n + 3] \]  
(C) \[ \delta[n] + 3\delta[n + 1] + 3\delta[n + 2] + \delta[n + 3] \]  
(D) \[ \delta[n] + 3\delta[n - 1] + 3\delta[n - 2] + \delta[n - 3] \]

MCQ 6.1.58 The time signal corresponding to \(z^6 + z^3 + 3 + 2z^{-1} + z^{-4}\), \(|z| > 0\) is

(A) \[ \delta[n + 6] + \delta[n + 2] + 3\delta[n] + 2\delta[n - 3] + \delta[n - 4] \]  
(B) \[ \delta[n - 6] + \delta[n - 2] + 3\delta[n] + 2\delta[n + 3] + \delta[n + 4] \]  
(C) \[ \delta[-n + 6] + \delta[-n + 2] + 3\delta[-n] + 2\delta[-n + 3] + \delta[-n + 4] \]  
(D) \[ \delta[-n - 6] + \delta[-n - 2] + 3\delta[-n] + 2\delta[-n - 3] + \delta[-n - 4] \]

MCQ 6.1.59 The time signal corresponding to \[ \frac{1}{1 - \frac{3}{4} z^{-1}} \], \(|z| > \frac{1}{2}\) is

(A) \[ 2^n, \quad n \text{ even} \]  
(B) \[ \left( \frac{1}{2} ight)^2 u[n] \]  
(C) \[ 2^n, \quad n \text{ odd}, n > 0 \]  
(D) \[ 2^n u[n] \]

MCQ 6.1.60 The time signal corresponding to \[ \frac{1}{1 - \frac{3}{4} z^{-1}} \], \(|z| < \frac{1}{2}\) is

(A) \[ -\sum_{k=0}^{\infty} 2^{2(k+1)} \delta[-n - 2(k + 1)] \]  
(B) \[ -\sum_{k=0}^{\infty} 2^{2(k+1)} \delta[-n + 2(k + 1)] \]  
(C) \[ -\sum_{k=0}^{\infty} 2^{2(k+1)} \delta[n + 2(k + 1)] \]  
(D) \[ -\sum_{k=0}^{\infty} 2^{2(k+1)} \delta[n - 2(k + 1)] \]

MCQ 6.1.61 The time signal corresponding to \(\ln(1 + z^{-1})\), \(|z| > 0\) is

(A) \[ \frac{(-1)^k}{k} \delta[n - k] \]  
(B) \[ \frac{(-1)^k}{k} \delta[n + k] \]  
(C) \[ \frac{(-1)^k}{k} \delta[n - k] \]  
(D) \[ \frac{(-1)^k}{k} \delta[n + k] \]

MCQ 6.1.62 \(X[z]\) of a system is specified by a pole zero pattern as following:

[Diagram of z-plane with marked points and axes]
Consider three different solution of $x[n]$

$$x_1[n] = \left[2^n - \left(\frac{1}{3}\right)^n\right]u[n]$$

$$x_2[n] = -2^n u[n-1] - \frac{1}{3^n} u[n]$$

$$x_3[n] = -2^n u[n-1] + \frac{1}{3^n} u[-n-1]$$

Correct solution is

(A) $x_1[n]$    (B) $x_2[n]$    (C) $x_3[n]$    (D) All three

MCQ 6.1.63

Consider three different signal

$$x_1[n] = \left[2^n - \left(\frac{1}{2}\right)^n\right]u[n]$$

$$x_2[n] = -2^n u[-n-1] + \frac{1}{2^n} u[-n-1]$$

$$x_3[n] = -2^n u[-n-1] - \frac{1}{2^n} u[n]$$

Following figure shows the three different region. Choose the correct for the ROC of signal

(A) $x_1[n]$    (B) $x_2[n]$    (C) $x_3[n]$    (D) $x_3[n]$    (E) $x_1[n]$    (F) $x_2[n]$    (G) $x_3[n]$

MCQ 6.1.64

Given the $z$-transform

$$X(z) = \frac{1 + \frac{1}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})}$$

For three different ROC consider there different solution of signal $x[n]$: 

(a) $|z| > \frac{1}{2}$, $x[n] = \left[\frac{1}{2^n-1} - \left(\frac{-1}{3}\right)^n\right]u[n]$

(b) $|z| < \frac{1}{3}$, $x[n] = \left[-\frac{1}{2^n-1} + \left(\frac{-1}{3}\right)^n\right]u[-n+1]$

(c) $\frac{1}{3} < |z| < \frac{1}{2}$, $x[n] = -\frac{1}{2^n-1} u[-n-1] - \left(\frac{-1}{3}\right)^n u[n]$

Correct solution are

(A) (a) and (b)    (B) (a) and (c)    (C) (b) and (c)    (D) (a), (b), (c)
MCQ 6.1.65
The X(z) has poles at \( z = \frac{1}{2} \) and \( z = -1 \). If \( x[1] = 1 \), \( x[-1] = 1 \), and the ROC includes the point \( z = \frac{3}{2} \). The time signal \( x[n] \) is

(A) \( \frac{1}{2^n} u[n] - (-1)^n u[-n - 1] \)  
(B) \( \frac{1}{2^n} u[n] - (-1)^n u[-n - 1] \)  
(C) \( \frac{1}{2^n} u[n] + u[-n + 1] \)  
(D) \( \frac{1}{2^n} u[n] + u[-n + 1] \)

MCQ 6.1.66
If \( x[n] \) is right-sided, \( X(z) \) has a signal pole and \( x[0] = 2 \), \( x[2] = \frac{1}{2} \), then \( x[n] \) is

(A) \( u[-n] \)  
(B) \( \frac{u[n]}{2^{n+1}} \)  
(C) \( u[-n] \)  
(D) \( a u[-n] \)  

MCQ 6.1.67
The \( z \)-transform of \( u[-n] \) is

(A) \( \frac{1}{1 - \frac{1}{2} z^{-1}} - \frac{1}{1 - \frac{1}{2} z^{-1}} \)  
(B) \( \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 - \frac{1}{2} z^{-1}} \)  
(C) \( \frac{1}{1 - \frac{1}{2} z^{-1}} - \frac{1}{1 - \frac{1}{2} z^{-1}} \)  
(D) \( \frac{1}{1 - \frac{1}{2} z^{-1}} \)

Common Data For Q. 68 - 73:
Given the \( z \)-transform pair \( x[n] \rightarrow \frac{z^2}{z^2 - 16}, \ |z| < 4 \)

MCQ 6.1.68
The \( z \)-transform of the signal \( x[n - 2] \) is

(A) \( \frac{z^4}{z^4 - 16} \)  
(B) \( \frac{(z + 2)^2}{(z + 2)^2 - 16} \)  
(C) \( \frac{1}{z^2 - 16} \)  
(D) \( \frac{(z - 2)^2}{(z - 2)^2 - 16} \)

MCQ 6.1.69
The \( z \)-transform of the signal \( y[n] = \frac{1}{2^n} x[n] \) is

(A) \( \frac{(z + 2)^2}{(z + 2)^2 - 16} \)  
(B) \( \frac{z^2}{z^2 - 4} \)  
(C) \( \frac{(z - 2)^2}{(z - 2)^2 - 16} \)  
(D) \( \frac{z^2}{z^2 - 64} \)

MCQ 6.1.70
The \( z \)-transform of the signal \( x[-n] \times x[n] \) is

(A) \( \frac{z^2}{16z^2 - 257z^4 - 16} \)  
(B) \( \frac{-16z^2}{(z^2 - 16)^2} \)  
(C) \( \frac{z^2}{257z^4 - 16z^4 - 16} \)  
(D) \( \frac{16z^2}{(z^2 - 16)^2} \)

MCQ 6.1.71
The \( z \)-transform of the signal \( nx[n] \) is

(A) \( \frac{32z^2}{(z^2 - 16)^2} \)  
(B) \( \frac{-32z^2}{(z^2 - 16)^2} \)  
(C) \( \frac{32z}{(z^2 - 16)^2} \)  
(D) \( \frac{-32z}{(z^2 - 16)^2} \)
MCQ 6.1.72
The z-transform of the signal $x[n+1] + x[n-1]$ is
(A) $\frac{(z+1)^2}{(z+1)^2 - 16} + \frac{(z-1)^2}{(z-1)^2 - 16}$
(B) $\frac{z(z^2 + 1)}{z^2 - 16}$
(C) $\frac{z^2(-1 + z)}{z^2 - 16}$
(D) None of the above

MCQ 6.1.73
The z-transform of the signal $x[n] \cdot x[n-3]$ is
(A) $\frac{z^3}{(z^2 - 16)^2}$
(B) $\frac{z^7}{(z^2 - 16)^2}$
(C) $\frac{z^5}{(z^2 - 16)^2}$
(D) None of the above

Common Data For Q. 74 - 78:
Given the z-transform pair $3^n u[n] \leftrightarrow X(z)$

MCQ 6.1.74
The time signal corresponding to $X(2z)$ is
(A) $n^3 3^n u[2n]$
(B) $\left(-\frac{3}{2}\right)^n n^2 u[n]$
(C) $\left(\frac{3}{2}\right)^n n^2 u[n]$
(D) $6^n n^2 u[n]$

MCQ 6.1.75
The time signal corresponding to $X(z^{-1})$ is
(A) $n^3 3^{-n} u[-n]$
(B) $n^3 3^{-n} u[-n]$
(C) $\frac{1}{n} 3^k u[n]$
(D) $\frac{1}{n} 3^k u[-n]$

MCQ 6.1.76
The time signal corresponding to $\frac{d}{dz} X(z)$ is
(A) $(n - 1) 3^n u[n-1]$
(B) $n^3 3^n u[n-1]$
(C) $(1-n) 3^n u[n-1]$
(D) $(n-1) 3^n u[n-1]$

MCQ 6.1.77
The time signal corresponding to $\left(\frac{z^2 - z^{-2}}{2}\right) X(z)$ is
(A) $\frac{1}{2} (x[n+2] - x[n-2])$
(B) $x[n+2] - x[n-2]$
(C) $\frac{1}{2} x[n-2] - x[n+2]$
(D) $x[n-2] - x[n+2]$

MCQ 6.1.78
The time signal corresponding to $\{X(z)^2\}$ is
(A) $x[n] x[n]$
(B) $x[n] x[n]$
(C) $x(n) x[-n]$
(D) $x[-n] x[-n]$

MCQ 6.1.79
A causal system has
Input, $x[n] = \delta[n] + \frac{1}{4} \delta[n-1] - \frac{1}{8} \delta[n-2]$ and
Output, $y[n] = \delta[n] - \frac{3}{4} \delta[n-1]$

The impulse response of this system is
(A) $\frac{1}{2} [5(-\frac{1}{2})^n - 2(\frac{1}{4})^n] u[n]$
(B) $\frac{1}{2} [5(\frac{1}{2})^n + 2(-\frac{1}{4})^n] u[n]$
(C) $\frac{1}{2} [5(\frac{1}{2})^n - 2(-\frac{1}{4})^n] u[n]$
(D) $\frac{1}{2} [5(\frac{1}{2})^n + 2(\frac{1}{4})^n] u[n]$

*Shipping Free*
MCQ 6.1.80
A causal system has
\[ x[n] = (-3)^n u[n] \]
\[ y[n] = [4(2)^n - (\frac{1}{2})^n] u[n] \]
The impulse response of this system is
(A) \( 7\left(\frac{1}{2}\right)^n - 10\left(\frac{1}{2}\right)^n \) \( u[n] \)
(B) \( 7(2^n) - 10\left(\frac{1}{2}\right)^n \) \( u[n] \)
(C) \( 10\left(\frac{1}{2}\right)^2 - 7(2)^n \) \( u[n] \)
(D) \( 10(2^n) - 7\left(\frac{1}{2}\right)^n \) \( u[n] \)

MCQ 6.1.81
A system has impulse response \( h[n] = (\frac{1}{2})^n u[n] \). The output \( y[n] \) to the input \( x[n] \) is given by \( y[n] = 2\delta[n - 4] \). The input \( x[n] \) is
(A) \( 2\delta[-n - 4] - \delta[-n - 5] \)
(B) \( 2\delta[n + 4] - \delta[n + 5] \)
(C) \( 2\delta[-n + 4] - \delta[-n - 5] \)
(D) \( 2\delta[n - 4] - \delta[n - 5] \)

MCQ 6.1.82
A system is described by the difference equation
The impulse response of the system is
(A) \( \delta[n] - 2\delta[n + 2] + 4\delta[n + 4] - 6\delta[n + 6] \)
(B) \( \delta[n] + 2\delta[n - 2] - 4\delta[n - 4] + 6\delta[n - 6] \)
(C) \( \delta[n] - \delta[n - 2] + \delta[n - 4] - \delta[n - 6] \)
(D) \( \delta[n] - \delta[n + 2] + \delta[n + 4] - \delta[n + 6] \)

MCQ 6.1.83
The impulse response of a system is given by \( h[n] = \frac{3}{4^n} u[n - 1] \). The difference equation representation for this system is
(A) \( 4y[n] - y[n - 1] = 3x[n - 1] \)
(B) \( 4y[n] - y[n + 1] = 3x[n + 1] \)
(C) \( 4y[n] + y[n - 1] = -3x[n - 1] \)
(D) \( 4y[n] + y[n + 1] = 3x[n + 1] \)

MCQ 6.1.84
The impulse response of a system is given by \( h[n] = \delta[n] - \delta[n - 5] \). The difference equation representation for this system is
(A) \( y[n] = x[n] - x[n - 5] \)
(B) \( y[n] = x[n] - x[n + 5] \)
(C) \( y[n] = x[n] + 5x[n - 5] \)
(D) \( y[n] = x[n] - 5x[n + 5] \)

MCQ 6.1.85
Consider the following three systems
\[ y_1[n] = 0.2y[n - 1] + x[n] - 0.3x[n - 1] + 0.02x[n - 2] \]
\[ y_2[n] = x[n] - 0.1x[n - 1] \]
\[ y_3[n] = 0.5y[n - 1] + 0.4x[n] - 0.3x[n - 1] \]
The equivalent system are
(A) \( y_1[n] \) and \( y_2[n] \)
(B) \( y_2[n] \) and \( y_3[n] \)
(C) \( y_1[n] \) and \( y_3[n] \)
(D) all
MCQ 6.1.86

The z-transform function of a stable system is 
\[ H(z) = \frac{2 - \frac{3}{2}z^{-1}}{(1 - 2z^{-1})(1 + \frac{1}{2}z^{-1})}. \]

The impulse response \( h[n] \) is

(A) \[ 2^n u[-n + 1] - \left(\frac{1}{2}\right)^n u[n] \]

(B) \[ -2^n u[-n - 1] + \left(-\frac{1}{2}\right)^n u[n] \]

(C) \[ -2^n u[-n - 1] - \left(-\frac{1}{2}\right)^n u[n] \]

(D) \[ 2^n u[n] - \left(\frac{1}{2}\right)^n u[n] \]

MCQ 6.1.87

The transfer function of a causal system is 
\[ H(z) = \frac{5z^2}{z^2 - z - 6}. \]

The impulse response is

(A) \[ (3^n + (-1)^n 2^{n+3}) u[n] \]

(B) \[ 2^{n+1} + 2(-2)^n u[n] \]

(C) \[ (3^n - (-1)^n 2^{n+3}) u[n] \]

(D) \[ 2^n u[n] - (\frac{1}{2})^n u[n] \]

MCQ 6.1.88

The transfer function of a system is given by 
\[ H(z) = \frac{z(3z - 2)}{z^2 - z - \frac{1}{4}}. \]

The system is

(A) causal and stable

(B) causal, stable and minimum phase

(C) minimum phase

(D) none of the above

MCQ 6.1.89

The z-transform of a signal \( x[n] \) is 
\[ X(z) = \frac{3}{1 - \frac{3}{8}z^{-1} + z^{-2}}. \]

If \( X(z) \) converges on the unit circle, \( x[n] \) is

(A) \[ \frac{1}{3^{n+3}} u[n] - \frac{3^{n+3}}{8} u[-n - 1] \]

(B) \[ \frac{1}{3^{n+1}} u[n] - \frac{3^{n+3}}{8} u[-n] \]

(C) \[ \frac{1}{3^{n+1}} u[n] - \frac{3^{n+3}}{8} u[-n] \]

(D) \[ \frac{1}{3^{n+3}} u[n] - \frac{3^{n+3}}{8} u[-n] \]

MCQ 6.1.90

The transfer function of a system is 
\[ H(z) = \frac{4z^{-1}}{(1 - \frac{1}{4}z^{-1})^2}, \quad |z| > \frac{1}{4}. \]

The \( h[n] \) is

(A) stable

(B) causal

(C) stable and causal

(D) none of the above

MCQ 6.1.91

The transfer function of a system is given as
\[ H(z) = \frac{2(z + \frac{1}{2})}{(z - \frac{1}{2})(z - \frac{1}{3})}. \]

Consider the two statements

Statement (1): System is causal and stable.

Statement (2): Inverse system is causal and stable.

The correct option is

(A) (1) is true

(B) (2) is true

(C) Both (1) and (2) are true

(D) Both are false
MCQ 6.1.92  The impulse response of the system shown below is

(A) \(2^{n-2}(1 - (-1)^{n})u[n] + \frac{1}{2}\delta[n]\)    
(B) \(\frac{2^n}{2}(1 + (-1)^{n})u[n] + \frac{1}{2}\delta[n]\)

(C) \(2^{n-2}(1 - (-1)^{n})u[n] - \frac{1}{2}\delta[n]\)    
(D) \(\frac{2^n}{2}[1 + (-1)^{n}]u[n] - \frac{1}{2}\delta[n]\)

MCQ 6.1.93  The system diagram for the transfer function \(H(z) = \frac{z}{z^2 + z + 1}\) is shown below.

The system diagram is a

(A) Correct solution
(B) Not correct solution
(C) Correct and unique solution
(D) Correct but not unique solution

************
EXERCISE 6.2

QUES 6.2.1 Consider a DT signal which is defined as follows
\[ x[n] = \begin{cases} \frac{1}{2^n}, & n \geq 0 \\ 0, & n < 0 \end{cases} \]
The z-transform of \( x[n] \) will be \( \frac{a^z}{az - 1} \) such that the value of \( a \) is _______.

QUES 6.2.2 If the z-transform of a sequence \( x[n] = \{1, 1, 1, \ldots\} \) is \( X(z) \), then what is the value of \( X(1/2) \)?

QUES 6.2.3 The z-transform of a discrete time signal \( x[n] \) is \( X(z) = \frac{z^2 + z}{z - 1} \). Then, \( x[0] + x[1] + x[2] = \) _______.

QUES 6.2.4 If \( x[n] = a^n u[n] \), then the z-transform of \( x[n+3]u[n] \) will be \( a^k \left( \frac{z}{z-k} \right) \), where \( k = \) _______.

QUES 6.2.5 The inverse z-transform of a function \( X(z) = \frac{z^3}{z-\alpha} \) is \( u[n-k] \) where the value of \( k \) is _______.

QUES 6.2.6 Let \( x[n] \rightarrow \frac{z}{z-\alpha} \rightarrow X(z) \) be a z-transform pair, where \( X(z) = \frac{z^2}{z - 3} \). What will be the value of \( x[5] \)?

QUES 6.2.7 The z-transform of a discrete time sequence \( y[n] = n[n+1]u[n] \) is \( \frac{kz^k}{(z-1)^k} \), such that the value of \( k \) is _______.

QUES 6.2.8 A signal \( x[n] \) has the following z-transform \( X(z) = \log(1 - 2z) \), \( \text{ROC: } |z| < \frac{1}{2} \). Let the signal be
\[ x[n] = \frac{1}{n} (\frac{1}{2})^n u[a - n] \]
what is the value of \( a \) in the expression?

QUES 6.2.9 Let \( x[n] \rightarrow \frac{z}{z-\alpha} \rightarrow X(z) \) be a z-transform pair. Consider another signal \( y[n] \) defined as
\[ y[n] = \begin{cases} x[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases} \]

The z-transform of \( y[n] \) is \( X(z^k) \) such that the value of \( k \) is _______.

QUES 6.2.10 Let \( X(z) \) be z-transform of a discrete time sequence \( x[n] = (-\frac{1}{2})^2 u[n] \). Consider another signal \( y[n] \) and its z-transform \( Y(z) \) given as \( Y(z) = X(z^2) \).

What is the value of \( y[n] \) at \( n = 4 \)?

QUES 6.2.11 Let \( h[n] = \{1, 2, 0, -1, 1\} \) and \( x[n] = \{1, 3, -1, -2\} \) be two discrete time sequences. What is the value of convolution \( y[n] = h[n] * x[n] \) at \( n = 4 \)?

QUES 6.2.12 A discrete time sequence is defined as follows
\[ x[n] = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{otherwise} \end{cases} \]

What is the final value of \( x[n] \)?

QUES 6.2.13 Let \( X(z) \) be the z-transform of a DT signal \( x[n] \) given as
\[ X(z) = \frac{0.5z^2}{(z-1)(z-0.5)} \]
Sample Chapter of Signals and Systems (Vol-7, GATE Study Package)

The initial value of $x[n]$ is _______

QUES 6.2.14

The signal $x[n] = (0.5)^n u[n]$ is when applied to a digital filter, it yields the following output $y[n] = \delta[n] - 2\delta[n-1]$. If impulse response of the filter is $h[n]$, then what will be the value of sample $h[1]$?

QUES 6.2.15

The transfer function for the system realization shown in the figure will be $k(z + 1) - 1$ such that the value of $k$ is _______

QUES 6.2.16

Consider a cascaded system shown in the figure

$X(z)$

where, $h_1[n] = \delta[n] + \frac{1}{2} \delta[n - 1]$ and $h_2[n] = \left(\frac{1}{z}\right)^n u[n]$.

If an input $x[n] = \cos(n\pi)$ is applied, then output $y[n] = k\cos(n\pi)$ where the constant $k$ is _______

QUES 6.2.17

The block diagram of a discrete time system is shown in the figure below

The system is BIBO stable for $|\alpha| < _______

QUES 6.2.18

Let $x[n] = \delta[n - 1] + \delta[n + 2]$. If unilateral z-transform of the signal $x[n]$ be $X(z) = z^k$ then, the value of constant $k$ is _______

QUES 6.2.19

The unilateral z-transform of signal $x[n] = u[n + 4]$ is $\frac{1}{1 + a/z}$ such that the value of $a$ is _______

QUES 6.2.20

If z-transform is given by $X(z) = \cos(z^3)$, $|z| > 0$, then what will be the value of $x[12]$?

QUES 6.2.21

The z-transform of an anticausal system is $X(z) = \frac{12 - 21z}{3 - 7z + 12z^2}$. What will be the value of $x[0]$?

QUES 6.2.22

The system $y[n] = cy[n-1] - 0.12y[n-2] + x[n-1] + x[n-2]$ is stable if $|c| < _______

***********
EXERCISE 6.3

MCQ 6.3.1 The z-transform is used to analyze
(A) discrete time signals and system
(B) continuous time signals and system
(C) both (A) and (B)
(D) none

MCQ 6.3.2 Which of the following expression is correct for the bilateral z-transform of $x[n]$?
(A) $\sum_{n=0}^{\infty} x[n]z^n$
(B) $\sum_{n=0}^{\infty} x[n]z^{-n}$
(C) $\sum_{n=0}^{\infty} x[n]z^n$
(D) $\sum_{n=0}^{\infty} x[n]z^{-n}$

MCQ 6.3.3 The unilateral z-transform of sequence $x[n]$ is defined as
(A) $\sum_{n=0}^{\infty} x[n]z^n$
(B) $\sum_{n=0}^{\infty} x[n]z^{-n}$
(C) $\sum_{n=0}^{\infty} x[n]z^{-n}$
(D) $\sum_{n=0}^{\infty} x[n]z^n$

MCQ 6.3.4 The z-transform of a causal signal $x[n]$ is given by
(A) $\sum_{n=-\infty}^{\infty} x[n]z^n$
(B) $\sum_{n=0}^{\infty} x[n]z^{-n}$
(C) $\sum_{n=0}^{\infty} x[n]z^{-n}$
(D) $\sum_{n=0}^{\infty} x[n]z^n$

MCQ 6.3.5 For a signal $x[n]$, its unilateral z-transform is equivalent to the bilateral z-transform of
(A) $x[n]r[n]$ (B) $x[n]e[n]$ (C) $x[n]u[n]$ (D) none of these

MCQ 6.3.6 The ROC of z-transform $X(z)$ is defined as the range of values of $z$ for which $X(z)$
(A) zero (B) diverges
(C) converges (D) none

MCQ 6.3.7 In the z-plane the ROC of z-transform $X(z)$ consists of a
(A) strip (B) parabola
(C) rectangle (D) ring

MCQ 6.3.8 If $x[n]$ is a right-sided sequence, and if the circle $z = r_0$ is in the ROC, then
(A) the values of $z$ for which $z > r_0$ will also be in the ROC
(B) the values of $z$ for which $z < r_0$ will also be in the ROC
(C) both (A) & (B)
(D) none of these

MCQ 6.3.9 The ROC does not contain any
(A) poles (B) 1's
(C) zeros (D) none

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MCQ 6.3.10 Let \( x[n] \xrightarrow{Z} X(z) \) be a z-transform pair. If \( x[n] = \delta[n] \), then the ROC of \( X(z) \) is
(A) \( |z| < 1 \)  
(B) \( |z| > 1 \)  
(C) entire z-plane  
(D) none of the above

MCQ 6.3.11 The ROC of z-transform of unit-step sequence \( u[n] \), is
(A) entire z-plane  
(B) \( |z| < 1 \)  
(C) \( |z| > 1 \)  
(D) none of the above

MCQ 6.3.12 The ROC of the unilateral z-transform of \( \alpha^n \) is
(A) \( |z| > |\alpha| \)  
(B) \( |z| < |\alpha| \)  
(C) \( |z| < 1 \)  
(D) \( |z| > 1 \)

MCQ 6.3.13 Which of the following statement about ROC is not true ?
(A) ROC never lies exactly at the boundary of a circle  
(B) ROC consists of a circle in the z-plane centred at the origin  
(C) ROC of a right handed finite sequence is the entire z-plane except \( z = 0 \)  
(D) ROC contains both poles and zeroes

MCQ 6.3.14 The z-transform of unit step sequence is
(A) 1  
(B) \( \frac{1}{z} \)  
(C) \( z^{-1} \)  
(D) 0

MCQ 6.3.15 The ROC for the z-transform of the sequence \( x[n] = u[-n] \) is
(A) \( |z| > 0 \)  
(B) \( |z| < 1 \)  
(C) \( |z| > 1 \)  
(D) does not exist

MCQ 6.3.16 Let \( x[n] \xrightarrow{Z} X(z) \), then unilateral z-transform of sequence \( x_1[n] = x[n-1] \) will be
(A) \( X_1(z) = z^{-1}X(z) + x[0] \)  
(B) \( X_1(z) = z^{-1}X(z) - x[1] \)  
(C) \( X_1(z) = z^{-1}X(z) - x[-1] \)  
(D) \( X_1(z) = z^{-1}X(z) + x[-1] \)

MCQ 6.3.17 Let \( x[n] \xrightarrow{Z} X(z) \), the bilateral z-transform of \( x[n - n_0] \) is given by
(A) \( zX(z) \)  
(B) \( z^nX(z) \)  
(C) \( z^{-n_0}X(z) \)  
(D) \( \frac{1}{z}X(z) \)

MCQ 6.3.18 If the ROC of z-transform of \( x[n] \) is \( R_x \) then the ROC of z-transform of \( x[-n] \) is
(A) \( R_x \)  
(B) \( -R_x \)  
(C) \( 1/R_x \)  
(D) none of these

MCQ 6.3.19 If \( X(z) = \mathcal{Z}\{x[n]\} \); then \( X(a^n) \) will be
(A) \( X(az) \)  
(B) \( X\left(\frac{z}{a}\right) \)  
(C) \( X\left(\frac{a}{z}\right) \)  
(D) \( X\left(\frac{1}{az}\right) \)
MCQ 6.3.20 If \( x[n] \) and \( y[n] \) are two discrete time sequences, then the \( z \)-transform of correlation of the sequences \( x[n] \) and \( y[n] \) is
(A) \( X(z)Y(z) \) 
(B) \( X(z)Y(z) \) 
(C) \( X(z)Y(z) \) 
(D) \( X(z)Y(z) \)

MCQ 6.3.21 If \( X(z) = \mathcal{Z}\{x[n]\} \), then, value of \( x[0] \) is equal to
(A) \( \lim_{z \to 0} z X(z) \) 
(B) \( \lim_{z \to 1} (z - 1) X(z) \) 
(C) \( \lim_{z \to 0} X(z) \) 
(D) \( \lim_{z \to 1} X(z) \)

MCQ 6.3.22 The choice of realization of structure depends on
(A) computational complexity 
(B) memory requirements 
(C) parallel processing and pipelining 
(D) all the above

MCQ 6.3.23 Which of the following schemes of system realization uses separate delays for input and output samples?
(A) parallel form 
(B) cascade form 
(C) direct form-I 
(D) direct form-II

MCQ 6.3.24 The direct form-I and II structures of IIR system will be identical in
(A) all pole system 
(B) all zero system 
(C) both (A) and (B) 
(D) first order and second order systems

MCQ 6.3.25 The number of memory locations required to realize the system,
\[
H(z) = \frac{1 + 3z^{-2} + 2z^{-3}}{1 + 2z^{-2} + z^{-4}}
\]
is
(A) 5 
(B) 7 
(C) 2 
(D) 10

MCQ 6.3.26 The mapping \( z = e^{sT} \) from \( s \)-plane to \( z \)-plane, is
(A) one to one 
(B) many to one 
(C) one to many 
(D) many to many

***********
EXERCISE 6.4

MCQ 6.4.1
What is the z-transform of the signal \( x[n] = a^n u[n] \)?

(A) \( X(z) = \frac{1}{z-1} \)

(B) \( X(z) = \frac{1}{1-z} \)

(C) \( X(z) = \frac{z}{z-\alpha} \)

(D) \( X(z) = \frac{1}{z-\alpha} \)

MCQ 6.4.2
The z-transform of the time function \( \sum_{k=0}^{\infty} \delta[n - k] \) is

(A) \( \frac{z-1}{z} \)

(B) \( \frac{z}{z-1} \)

(C) \( \frac{z}{(z-1)^2} \)

(D) \( \frac{(z-1)^2}{z} \)

MCQ 6.4.3
The z-transform \( F(z) \) of the function \( f(nT) = a^{nT} \) is

(A) \( \frac{z}{z-a} \)

(B) \( \frac{z}{z+a} \)

(C) \( \frac{z}{z-a^2} \)

(D) \( \frac{z}{z+a^2} \)

MCQ 6.4.4
The discrete-time signal \( x[n] = \sum_{n=0}^{\infty} \frac{2^n z^{-n}}{z^{n+1}} \) is orthogonal to the signal

(A) \( y_1[n] = \sum_{n=0}^{\infty} \frac{3^n z^{-n}}{z^{n+1}} \)

(B) \( y_2[n] = \sum_{n=0}^{\infty} \frac{5^n z^{-n} - n}{z^{n+1}} \)

(C) \( y_3[n] = \sum_{n=0}^{\infty} 2^n \frac{z^{-n}}{z^{n+1}} \)

(D) \( y_4[n] = \sum_{n=0}^{\infty} 3^n \frac{z^{-n}}{z^{n+1}} \)

MCQ 6.4.5
Which one of the following is the region of convergence (ROC) for the sequence \( x[n] = b^n u[n] + b^{-n} u[-n-1] \); \(|b| < 1\)?

(A) Region \( |z| < 1 \)

(B) Annular strip in the region \( b > |z| > \frac{1}{b} \)

(C) Region \( |z| > 1 \)

(D) Annular strip in the region \( b < |z| < \frac{1}{b} \)

MCQ 6.4.6
Assertion (A) : The signals \( a^n u[n] \) and \(-a^n u[-n-1]\) have the same z-transform, \( z/(z-a) \).

Reason (R) : The Region of Convergence (ROC) for \( a^n u[n] \) is \( |z| > |a| \), whereas the ROC for \( a^n u[-n-1] \) is \( |z| < |a| \).

(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true but R is NOT the correct explanation of A

(C) A is true but R is false

(D) A is false but R is true

MCQ 6.4.7
Which one of the following is the correct statement?

The region of convergence of z-transform of \( x[n] \) consists of the values of \( z \) for which \( x[n] r^{-n} \) is

(A) absolutely integrable

(B) absolutely summable

(C) unity

(D) \(< 1 \)
The ROC of z-transform of the sequence $x[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n - 1]$ is

(A) $\frac{1}{3} < |z| < \frac{1}{2}$  
(B) $|z| > \frac{1}{2}$

(C) $|z| < \frac{1}{3}$  
(D) $2 < |z| < 3$

The region of convergence of $z$-transform of the sequence $(\frac{5}{6})^n u[n] - (\frac{6}{5})^n u[-n - 1]$ must be

(A) $|z| < \frac{5}{6}$  
(B) $|z| > \frac{5}{6}$

(C) $\frac{5}{6} < |z| < \frac{6}{5}$  
(D) $\frac{6}{5} < |z| < \infty$

The region of convergence of the $z$-transform of the discrete-time signal $x[n] = z^n u[n]$ will be

(A) $|z| > 2$  
(B) $|z| < 2$

(C) $|z| > \frac{1}{2}$  
(D) $|z| < \frac{1}{2}$

The region of convergence of the $z$-transform of a unit step function is

(A) $|z| > 1$  
(B) $|z| < 1$

(C) (Real part of $z$) > 0  
(D) (Real part of $z$) < 0

Match List I (Discrete Time signal) with List II (Transform) and select the correct answer using the codes given below the lists :

<table>
<thead>
<tr>
<th>List I</th>
<th>List II</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Unit step function</td>
<td>1. $\frac{z - \cos \omega T}{z^2 - 2z \cos \omega T + 1}$</td>
</tr>
<tr>
<td>B. Unit impulse function</td>
<td>2. $\frac{z}{z - 1}$</td>
</tr>
<tr>
<td>C. $\sin \omega t$, $t = 0, T, 2T$</td>
<td>3. $\frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$</td>
</tr>
<tr>
<td>D. $\cos \omega t$, $t = 0, T, 2T$, ......</td>
<td>4. $\frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$</td>
</tr>
</tbody>
</table>

Codes :

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>(B)</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>(C)</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>(D)</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

What is the inverse z-transform of $X(z)$

(A) $\frac{1}{2\pi j} \int X(z) z^{n-1} dz$  
(B) $2\pi j \int X(z) z^{n+1} dz$

(C) $\frac{1}{2\pi j} \int X(z) z^{-n} dz$  
(D) $2\pi j \int X(z) z^{-(n+1)} dz$

Which one of the following represents the impulse response of a system defined by $H(z) = z^{-m}$?

(A) $u[n - m]$  
(B) $\delta[n - m]$

(C) $\delta[n]$  
(D) $\delta[m - n]$

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MCQ 6.4.15
If \( X(z) \) is \( \frac{1}{|1-z^{-1}|} \) with \( |z| > 1 \), then what is the corresponding \( x[n] \)?

(A) \( e^{-n} \)  
(B) \( e^n \)  
(C) \( u[n] \)  
(D) \( \delta(n) \)

MCQ 6.4.16
The \( z \)-transform \( X(z) \) of a sequence \( x[n] \) is given by \( X[z] = \frac{0.5}{1 - z} \). It is given that the region of convergence of \( X(z) \) includes the unit circle. The value of \( x[0] \) is

(A) \(-0.5\)  
(B) \(0\)  
(C) \(0.25\)  
(D) \(0.5\)

MCQ 6.4.17
If \( u(t) \) is the unit step and \( \delta(t) \) is the unit impulse function, the inverse \( z \)-transform of \( F(z) = \frac{1}{z-1} \) for \( k > 0 \) is

(A) \(-(-1)^k \delta(k)\)  
(B) \(\delta(k) - (-1)^k\)  
(C) \((-1)^k u(k)\)  
(D) \(u(k) - (-1)^k\)

MCQ 6.4.18
For a \( z \)-transform \( X(z) = \frac{2z - \frac{5}{6}}{(z - \frac{1}{2})(z - \frac{1}{3})} \), match List I (The sequences) with List II (The region of convergence) and select the correct answer using the codes given below the lists:

<table>
<thead>
<tr>
<th>List I</th>
<th>List II</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. ((1/2)^n + (1/3)^n)u[n]</td>
<td>1. (</td>
</tr>
<tr>
<td>B. ((1/2)^nu[n] - (1/3)^nu[-n - 1])</td>
<td>2. (</td>
</tr>
<tr>
<td>C. (-((1/2)^n + (1/3)^n)u[-n - 1])</td>
<td>3. (</td>
</tr>
<tr>
<td>D. ((- ((1/2)^n + (1/3)^n)u[-n - 1])</td>
<td>4. (</td>
</tr>
</tbody>
</table>

Codes:

(A) 4 2 1 3
(B) 4 2 3 1
(C) 4 3 2 1
(D) 3 4 2 1

MCQ 6.4.19
Which one of the following is the inverse \( z \)-transform of \( X(z) = \frac{z}{(z - 2)(z - 3)^2} \) for \( |z| < 2 \)?

(A) \([2^n - 3^n]u[-n - 1]\)  
(B) \([3^n - 2^n]u[-n - 1]\)  
(C) \([2^n - 3^n]u[n + 1]\)  
(D) \([2^n - 3^n]u[n]\)

MCQ 6.4.20
Given \( X(z) = \frac{z}{(z - a)^3} \) with \( |z| > a \), the residue of \( X(z)z^{-1} \) at \( z = a \) for \( n \geq 0 \) will be

(A) \(a^n\)  
(B) \(a^{-n}\)  
(C) \(na^n\)  
(D) \(na^{-n}\)

MCQ 6.4.21
Given \( X(z) = \frac{1}{1 - az^{-1}} + \frac{1}{1 - bz^{-1}} \), then \( x[0] \) of the corresponding sequence is given by

(A) \(\frac{1}{2}\)  
(B) \(\frac{5}{6}\)  
(C) \(\frac{1}{2}\)  
(D) \(\frac{1}{6}\)
The Z-Transform

If \( X(z) = \frac{z + z^{-3}}{z + z^{-1}} \) then \( x[n] \) series has

(A) alternate 0’s  
(B) alternate 1’s  
(C) alternate 2’s  
(D) alternate -1’s

Consider the z-transform \( x(z) = 5z^2 + 4z^{-1} + 3; 0 < |z| < \infty \). The inverse z-transform \( x[n] \) is

(A) \( 5\delta[n + 2] + 3\delta[n] + 4\delta[n - 1] \)  
(B) \( 5\delta[n - 2] + 3\delta[n] + 4\delta[n + 1] \)  
(C) \( 5u[n + 2] + 3u[n] + 4u[n - 1] \)  
(D) \( 5u[n - 2] + 3u[n] + 4u[n + 1] \)

The sequence \( x[n] \) whose z-transform is \( X(z) = e^{j\omega} \) is

(A) \( (-1)^n \frac{1}{n!} u[-n] \)  
(B) \( \frac{1}{(n+1)!} u[-n-1] \)  
(C) \( \alpha^n u[n] \)  
(D) \( \frac{1}{(1-\alpha z^{-1})} \)

If the region of convergence of \( x_1[n] + x_2[n] \) is \( \frac{1}{3} < |z| < \frac{2}{3} \) then the region of convergence of \( x_1[n] - x_2[n] \) includes

(A) \( \frac{1}{3} < |z| < 3 \)  
(B) \( \frac{2}{3} < |z| < 3 \)  
(C) \( \frac{2}{3} < |z| < 3 \)  
(D) \( \frac{1}{3} < |z| < \frac{2}{3} \)

Match List I with List II and select the correct answer using the codes given below the lists:

List I  
A. \( \alpha^n u[n] \)  
B. \( -\alpha^n u[-n-1] \)  
C. \( -n\alpha^n u[-n-1] \)  
D. \( n\alpha^n u[n] \)

Codes:

(A) 2 4 3 1  
(B) 1 3 4 2  
(C) 1 4 3 2  
(D) 2 3 4 1

Algebraic expression for z-transform of \( x[n] \) is \( X[z] \). What is the algebraic expression for z-transform of \( \{e^{j\omega n} x[n]\} \) ?

(A) \( x(z - z_0) \)  
(B) \( e^{j\omega} X(z) \)  
(C) \( x(e^{j\omega}) \)  
(D) \( X(z) e^{j\omega z} \)

Given that \( F(z) \) and \( G(z) \) are the one-sided z-transforms of discrete time functions \( f(nT) \) and \( g(nT) \), the z-transform of \( \sum f(kT)g(nT - kT) \) is given by

(A) \( \sum f(nT)g(nT)z^n \)  
(B) \( \sum f(nT)z^n \sum g(nT)z^{-n} \)  
(C) \( \sum f(kT)g(nT - kT)z^n \)  
(D) \( \sum f(nT - kT)g(nT)z^{-n} \)
Sample Chapter of Signals and Systems (Vol-7, GATE Study Package)

MCQ 6.4.29  
IES E & T 1997  
Match List-I \((x[n])\) with List-II \((X(z))\) and select the correct answer using the codes given below the Lists:

**List-I**

A. \(a^n u[n]\)

B. \(a^{n-2} u[n - 2]\)

C. \(e^{\alpha n} a^n\)

D. \(na^n u[n]\)

**List-II**

1. \(\frac{az}{(z - a)^2}\)

2. \(\frac{z e^{-j}}{z e^{-j} - a}\)

3. \(\frac{z}{z - a}\)

4. \(\frac{z^{-1}}{z - a}\)

**Codes:**

A B C D

(A) 3 2 4 1

(B) 2 3 4 1

(C) 3 4 2 1

(D) 1 4 2 3

---

MCQ 6.4.30  
IES EC 2005  
The output \(y[n]\) of a discrete time LTI system is related to the input \(x[n]\) as given below:

\[y[n] = \sum_{k=0}^{\infty} x[k]\]

Which one of the following correctly relates the \(z\)-transform of the input and output, denoted by \(X(z)\) and \(Y(z)\), respectively?

(A) \(Y(z) = (1 - z^{-1})X(z)\)

(B) \(Y(z) = z^{-1}X(z)\)

(C) \(Y(z) = \frac{X(z)}{1 - z^{-1}}\)

(D) \(Y(z) = \frac{dX(z)}{dz}\)

---

MCQ 6.4.31  
IES EC 2010  
Convolution of two sequence \(x_1[n]\) and \(x_2[n]\) is represented as

(A) \(X_1(z) \ast X_2(z)\)

(B) \(X_1(z) X_2(z)\)

(C) \(X_1(z) + X_2(z)\)

(D) \(X_1(z)/X_2(z)\)

---

MCQ 6.4.32  
GATE EC 1999  
The \(z\)-transform of a signal is given by \(C(z) = \frac{1z^{-1}(1 - z^{-1})}{4(1 - z^{-1})^2}\). Its final value is

(A) 1/4

(B) zero

(C) 1.0

(D) infinity

---

MCQ 6.4.33  
IES E & T 1996  
Consider a system described by the following difference equation:

\[y(n + 3) + 6y(n + 2) + 11y(n + 1) + 6y(n) = r(n + 2) + 9r(n + 1) + 20r(n)\]

where \(y\) is the output and \(r\) is the input. The transfer function of the system is

\[
\begin{align*}
\text{(A)} & \quad \frac{2z^2 + z + 20}{3z^2 + 2z^2 + z + 6} \\
\text{(B)} & \quad \frac{z^2 + 9z + 20}{z^3 + 6z^2 + 6z + 11} \\
\text{(C)} & \quad \frac{z^3 + 6z^2 + 6z + 11}{z^2 + 9z + 20}
\end{align*}
\]

(D) none of the above

---

MCQ 6.4.34  
IES E & T 1998  
If the function \(H_1(z) = (1 + 1.5z^{-1} - z^{-2})\) and \(H_2(z) = z^2 + 1.5z - 1\), then

(A) the poles and zeros of the functions will be the same

(B) the poles of the functions will be identical but not zeros

(C) the zeros of the functions will be identical but not the poles

(D) neither the poles nor the zeros of the two functions will be identical
The state model

\[
\begin{align*}
    x[k+1] &= \begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k] \\
y[k] &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix}
\end{align*}
\]

is represented in the difference equation as

(A) \( c[k+2] + \alpha c[k+1] + \beta c[k] = u[k] \)
(B) \( c[k+1] + \alpha c[k] + \beta c[k-1] = u[k-1] \)
(C) \( c[k-2] + \alpha c[k-1] + \beta c[k] = u[k] \)
(D) \( c[k-1] + \alpha c[k] + \beta c[k+1] = u[k+1] \)

MCQ 6.4.36
The impulse response of a discrete system with a simple pole shown in the figure below. The pole of the system must be located on the

(A) real axis at \( z = -1 \)
(B) real axis between \( z = 0 \) and \( z = 1 \)
(C) imaginary axis at \( z = j \)
(D) imaginary axis between \( z = 0 \) and \( z = j \)

MCQ 6.4.37
Which one of the following digital filters does have a linear phase response?

(A) \( y[n] + y[n-1] = x[n] - x[n-1] \)
(B) \( y[n] = \frac{1}{6}(3x[n] + 2x[n-1] + x[n-2]) \)
(C) \( y[n] = \frac{1}{6}(x[n] + 2x[n-1] + 3x[n-2]) \)
(D) \( y[n] = \frac{1}{4}(x[n] + 2x[n-1] + x[n-2]) \)

MCQ 6.4.38
The poles of a digital filter with linear phase response can lie

(A) only at \( z = 0 \)
(B) only on the unit circle
(C) only inside the unit circle but not at \( z = 0 \)
(D) on the left side of \( \text{Real}(z) = 0 \) line

MCQ 6.4.39
The impulse response of a discrete system with a simple pole is shown in the given figure

(A) real axis at \( z = -1 \)
(B) real axis between \( z = 0 \) and \( z = 1 \)
(C) imaginary axis at \( z = j \)
(D) imaginary axis between \( z = 0 \) and \( z = j \)
The pole must be located
(A) on the real axis at \( z = 1 \)    (B) on the real axis at \( z = -1 \)
(C) at the origin of the \( z \)-plane    (D) at \( z = \infty \)

MCQ 6.4.40

The response of a linear, time-invariant discrete-time system to a unit step input \( u[n] \) is the unit impulse \( \delta[n] \). The system response to a ramp input \( nu[n] \) would be
(A) \( u[n] \)    (B) \( u[n-1] \)
(C) \( n\delta[n] \)    (D) \( \sum_{k=0}^{\infty} k\delta[n-k] \)

MCQ 6.4.41

A system can be represented in the form of state equations as
\[
s[n+1] = As[n] + Bx[n]
y[n] = Cs[n] + Dx[n]
\]
where \( A, B, C \) and \( D \) are matrices, \( s[n] \) is the state vector, \( x[n] \) is the input and \( y[n] \) is the output. The transfer function of the system \( H(z) = Y(z)/X(z) \) is given by
(A) \( A(zI - B)^{-1}C + D \)    (B) \( B(zI - C)^{-1}D + A \)
(C) \( C(zI - A)^{-1}B + D \)    (D) \( D(zI - A)^{-1}C + B \)

MCQ 6.4.42

What is the number of roots of the polynomial \( F(z) = 4z^2 - 8z^2 - z + 2 \), lying outside the unit circle?
(A) 0    (B) 1
(C) 2    (D) 3

MCQ 6.4.43

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k] \]

Which one of the following systems is inverse of the system given above?
(A) \( x[n] = y[n] - y[n-1] \)    (B) \( x[n] = y[n] \)
(C) \( x[n] = y[n] + 4 \)    (D) \( x[n] = ny[n] \)

MCQ 6.4.44

For the system shown, \( x[n] = k\delta[n] \), and \( y[n] \) is related to \( x[n] \) as
\[ y[n] = \frac{1}{2}y[n-1] = x[n] \]

What is \( y[n] \) equal to?
(A) \( k \)    (B) \( \left(\frac{1}{2}\right)^n k \)
(C) \( nk \)    (D) \( 2^n \)

MCQ 6.4.45

Unit step response of the system described by the equation
\[ y[n] + y[n-1] = x[n] \]
is
(A) \( \frac{z^2}{(z+1)(z-1)} \)    (B) \( \frac{z}{(z+1)(z-1)} \)
(C) \( \frac{z+1}{z-1} \)    (D) \( \frac{z(z-1)}{(z+1)} \)

MCQ 6.4.46

Unit step response of the system described by the equation
\[ y[n] + y[n-1] = x[n] \]
is
(A) \( \frac{z^2}{(z+1)(z-1)} \)    (B) \( \frac{z}{(z+1)(z-1)} \)
(C) \( \frac{(z+1)}{(z-1)} \)    (D) \( \frac{z(z-1)}{(z+1)} \)
The Z-Transform

MCQ 6.4.47
System transformation function $H(z)$ for a discrete time LTI system expressed in state variable form with zero initial conditions is
(A) $c(zI - A)^{-1}b + d$
(B) $c(zI - A)^{-1}$
(C) $(zI - A)^{-1}$
(D) $(zI - A)^{-1}z$

MCQ 6.4.48
A system with transfer function $H(z)$ has impulse response $h(.)$ defined as $h(2)=1, h(3) = -1$ and $h(k) = 0$ otherwise. Consider the following statements.

S1: $H(z)$ is a low-pass filter.
S2: $H(z)$ is an FIR filter.

Which of the following is correct?
(A) Only S2 is true
(B) Both S1 and S2 are false
(C) Both S1 and S2 are true, and S2 is a reason for S1
(D) Both S1 and S2 are true, but S2 is not a reason for S1

MCQ 6.4.49
The z-transform of a system is $X(z) = \frac{1}{z^2}$. If the ROC is $|z| > 0.2$, then the impulse response of the system is
(A) $6$ 
(B) zero
(C) $2$
(D) $-4$

MCQ 6.4.51
A sequence $x[n]$ with the z-transform $x(z) = z^4 + z^2 - 2z + 2 - 3z^{-4}$ is applied as an input to a linear, time-invariant system with the impulse response $h[n] = 2\delta[n-3]$ where $x[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$

The output at $n = 4$ is
(A) $6$
(B) zero
(C) $2$
(D) $-4$

MCQ 6.4.52
$H(z)$ is a transfer function of a real system. When a signal $x[n] = (1+j)^n$ is the input to such a system, the output is zero. Further, the Region of convergence (ROC) of $(1 - \frac{1}{2}z^{-1}) H(z)$ is the entire Z-plane (except $z = 0$). It can then be inferred that $H(z)$ can have a minimum of
(A) one pole and one zero
(B) one pole and two zeros
(C) two poles and one zero
(D) two poles and two zeros
A discrete-time signal, \( x[n] \), suffered a distortion modeled by an LTI system with \( H(z) = (1 - az^{-1}) \), \( a \) is real and \( |a| > 1 \). The impulse response of a stable system that exactly compensates the magnitude of the distortion is

(A) \( \left( \frac{1}{a} \right)^n u[n] \)
(B) \( -\left( \frac{1}{a} \right)^n u[-n-1] \)
(C) \( a^n u[n] \)
(D) \( a^n u[-n-1] \)

**MCQ 6.4.54**

**Assertion (A)**: A linear time-invariant discrete-time system having the system function

\[ H(z) = \frac{z}{z + \frac{1}{2}} \]

is a stable system.

**Reason (R)**: The pole of \( H(z) \) is in the left-half plane for a stable system.
(A) Both A and R are true and R is the correct explanation of A
(B) Both A and R are true but R is NOT a correct explanation of A
(C) A is true but R is false
(D) A is false but R is true

**MCQ 6.4.55**

**Assertion (A)**: An LTI discrete system represented by the difference equation

\[ y[n + 2] - 5y[n + 1] + 6y[n] = x[n] \]

is unstable.

**Reason (R)**: A system is unstable if the roots of the characteristic equation lie outside the unit circle.
(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is NOT the correct explanation of A.
(C) A is true but R is false.
(D) A is false but R is true.

**MCQ 6.4.56**

Consider the following statements regarding a linear discrete-time system

\[ H(z) = \frac{z^2 + 1}{(z + 0.5)(z - 0.5)} \]

1. The system is stable
2. The initial value \( h(0) \) of the impulse response is \(-4\)
3. The steady-state output is zero for a sinusoidal discrete time input of frequency equal to one-fourth the sampling frequency.

Which of these statements are correct?
(A) 1, 2 and 3
(B) 1 and 2
(C) 1 and 3
(D) 2 and 3

**MCQ 6.4.57**

**Assertion (A)**: The discrete-time system described by \( y[n] = 2x[n] + 4x[n - 1] \) is unstable, (here \( y[n] \) is the output and \( x[n] \) the input).

**Reason (R)**: It has an impulse response with a finite number of non-zero samples.
(A) Both A and R are true and R is the correct explanation of A
(B) Both A and R are true but R is NOT the correct explanation of A
(C) A is true but R is false
(D) A is false but R is true

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MCQ 6.4.58  
If the impulse response of discrete-time system is 
\[ h[n] = -5^u[-n - 1], \]
then the system function \( H(z) \) is equal to
(A) \( \frac{-z}{z - 5} \) and the system is stable
(B) \( \frac{z}{z - 5} \) and the system is stable
(C) \( \frac{-z}{z - 5} \) and the system is unstable
(D) \( \frac{z}{z - 5} \) and the system is unstable

MCQ 6.4.59  
H(z) is a discrete rational transfer function. To ensure that both \( H(z) \) and its inverse are stable its
(A) poles must be inside the unit circle and zeros must be outside the unit circle.
(B) poles and zeros must be inside the unit circle.
(C) poles and zeros must be outside the unit circle.
(D) poles must be outside the unit circle and zeros should be inside the unit circle.

MCQ 6.4.60  
**Assertion (A)**: The stability of the system is assured if the Region of Convergence (ROC) includes the unit circle in the \( z \)-plane.
**Reason (R)**: For a causal stable system all the poles should be outside the unit circle in the \( z \)-plane.
(A) Both A and R are true and R is the correct explanation of A
(B) Both A and R are true but R is NOT the correct explanation of A.
(C) A is true but R is false
(D) A is false but R is true

MCQ 6.4.61  
**Assertion (A)**: For a rational transfer function \( H(z) \) to be causal, stable and causally invertible, both the zeros and the poles should lie within the unit circle in the \( z \)-plane.
**Reason (R)**: For a rational system, ROC bounded by poles.
(A) Both A and R are true and R is the correct explanation of A
(B) Both A and R are true but R is NOT the correct explanation of A.
(C) A is true but R is false
(D) A is false but R is true

MCQ 6.4.62  
The transfer function of a discrete time LTI system is
\[ H(z) = \frac{2 - \frac{3}{4}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \]
Consider the following statements:
S1: The system is stable and causal for ROC: \( |z| > 1/2 \)
S2: The system is stable but not causal for ROC: \( |z| < 1/4 \)
S3: The system is neither stable nor causal for ROC: \( 1/4 < |z| < 1/2 \)
Which one of the following statements is valid?
(A) Both S1 and S2 are true
(B) Both S2 and S3 are true
(C) Both S1 and S3 are true
(D) S1, S2 and S3 are all true

MCQ 6.4.63  
A causal LTI system is described by the difference equation
\[ 2y[n] = \alpha y[n - 2] - 2x[n] + \beta x[n - 1] \]
The system is stable only if
(A) \( |\alpha| = 2, |\beta| < 2 \)
(B) \( |\alpha| > 2, |\beta| < 2 \)
(C) \( |\alpha| < 2, \) any value of \( \beta \)
(D) \( |\beta| < 2, \) any value of \( \alpha \)
MCQ 6.4.64
Two linear time-invariant discrete time systems $s_1$ and $s_2$ are cascaded as shown in the figure below. Each system is modelled by a second order difference equation. The difference equation of the overall cascaded system can be of the order of

(A) 0, 1, 2, 3 or 4
(B) either 2 or 4
(C) 2
(D) 4

MCQ 6.4.65
Consider the compound system shown in the figure below. Its output is equal to the input with a delay of two units. If the transfer function of the first system is given by

$$H_1(z) = \frac{z - 0.5}{z - 0.8},$$

then the transfer function of the second system would be

(A) $H_2(z) = \frac{z^2 - 0.2z^3}{1 - 0.4z^{-1}}$
(B) $H_2(z) = \frac{z^2 - 0.8z^3}{1 - 0.5z^{-1}}$
(C) $H_2(z) = \frac{z^2 - 0.2z^3}{1 - 0.4z^{-1}}$
(D) $H_2(z) = \frac{z^2 + 0.8z^3}{1 + 0.5z^{-1}}$

MCQ 6.4.66
Two systems $H_1(z)$ and $H_2(z)$ are connected in cascade as shown below. The overall output $y[n]$ is the same as the input $x[n]$ with a one unit delay. The transfer function of the second system $H_2(z)$ is

$$x(n) \rightarrow H_1(z) = \frac{(1 - 0.4z^{-1})}{(1 - 0.6z^{-1})} \rightarrow H_2(z) \rightarrow y(n)$$

(A) $\frac{1 - 0.6z^{-1}}{z^{-1}(1 - 0.4z^{-1})}$
(B) $\frac{z^{-1}(1 - 0.6z^{-1})}{(1 - 0.4z^{-1})}$
(C) $\frac{z^{-1}(1 - 0.4z^{-1})}{(1 - 0.6z^{-1})}$
(D) $\frac{1 - 0.4z^{-1}}{z^{-1}(1 - 0.6z^{-1})}$

MCQ 6.4.67
Two discrete time systems with impulse response $h_1[n] = \delta[n - 1]$ and $h_2[n] = \delta[n - 2]$ are connected in cascade. The overall impulse response of the cascaded system is

(A) $\delta[n - 1] + \delta[n - 2]$
(B) $\delta[n - 4]$
(C) $\delta[n - 3]$
(D) $\delta[n - 1] \delta[n - 2]$

MCQ 6.4.68
A cascade of three Linear Time Invariant systems is causal and unstable. From this, we conclude that

(A) each system in the cascade is individually causal and unstable
(B) at least on system is unstable and at least one system is causal
(C) at least one system is causal and all systems are unstable
(D) the majority are unstable and the majority are causal
MCQ 6.4.69
The minimum number of delay elements required in realizing a digital filter with the transfer function
\[ H(z) = \frac{1 + az^{-1} + bz^{-2}}{1 + cz^{-1} + dz^{-2} + ez^{-3}} \]
(A) 2 (B) 3 (C) 4 (D) 5

MCQ 6.4.70
A direct form implementation of an LTI system with
\[ H(z) = \frac{1}{1 - 0.7z^{-1} + 0.13z^{-2}} \]
is shown in figure. The value of \( a_0 \), \( a_1 \) and \( a_2 \) are respectively
(A) 1.0, 0.7 and −0.13 (B) −0.13, 0.7 and 1.0 (C) 1.0, −0.7 and 0.13 (D) 0.13, −0.7 and 1.0

MCQ 6.4.71
A digital filter having a transfer function
\[ H(z) = \frac{\rho_0 + \rho_1 z^{-1} + \rho_2 z^{-3}}{1 + d_3 z^{-3}} \]
is implemented using Direct Form-I and Direct Form-II realizations of IIR structure. The number of delay units required in Direct Form-I and Direct Form-II realizations are, respectively
(A) 6 and 6 (B) 6 and 3 (C) 3 and 3 (D) 3 and 2

MCQ 6.4.72
Consider the discrete-time system shown in the figure where the impulse response of \( G(z) \) is \( g(0) = 0, g(1) = g(2) = 1, g(3) = g(4) = \ldots = 0 \)
This system is stable for range of values of \( K \)
(A) \([−1, \frac{1}{2}]\) (B) \([−1, 1]\) (C) \([-\frac{1}{2}, 1] \) (D) \([-\frac{1}{2}, 2]\)

MCQ 6.4.73
In the IIR filter shown below, \( a \) is a variable gain. For which of the following cases, the system will transit from stable to unstable condition?
(A) \( 0.1 < a < 0.5 \) (B) \( 0.5 < a < 1.5 \) (C) \( 1.5 < a < 2.5 \) (D) \( 2 < a < \infty \)
The poles of an analog system are related to the corresponding poles of the digital system by the relation \( z = e^{sT} \). Consider the following statements.

1. Analog system poles in the left half of \( s \)-plane map onto digital system poles inside the circle \( |z| = 1 \).
2. Analog system zeros in the left half of \( s \)-plane map onto digital system zeros inside the circle \( |z| = 1 \).
3. Analog system poles on the imaginary axis of \( s \)-plane map onto digital system zeros on the unit circle \( |z| = 1 \).
4. Analog system zeros on the imaginary axis of \( s \)-plane map onto digital system zeros on the unit circle \( |z| = 1 \).

Which of these statements are correct?

(A) 1 and 2
(B) 1 and 3
(C) 3 and 4
(D) 2 and 4

MCQ 6.4.75

Which one of the following rules determines the mapping of \( s \)-plane to \( z \)-plane?

(A) Right half of the \( s \)-plane maps into outside of the unit circle in \( z \)-plane
(B) Left half of the \( s \)-plane maps into inside of the unit circle
(C) Imaginary axis in \( s \)-plane maps into the circumference of the unit circle
(D) All of the above

MCQ 6.4.76

Assertion (A) : The \( z \)-transform of the output of an ideal sampler is given by

\[ Z[f(t)] = K_0 + \frac{K_1}{z} + \frac{K_2}{z^2} + \cdots + \frac{K_n}{z^n} \]

Reason (R) : The relationship is the result of application of \( z = e^{-sT} \), where \( T \) stands for the time gap between the samples.

(A) Both A and R are true and R is the correct explanation of A
(B) Both A and R are true but R is NOT the correct explanation of A
(C) A is true but R is false
(D) A is false but R is true

MCQ 6.4.77

\( z \) and Laplace transform are related by

(A) \( s = \ln z \)
(B) \( s = \frac{\ln z}{1} \)
(C) \( s = z \)
(D) \( \frac{T}{\ln z} \)

MCQ 6.4.78

Frequency scaling [relationship between discrete time frequency (\( \Omega \)) and continuous time frequency (\( \omega \))] is defined as

(A) \( \omega = 2\Omega \)
(B) \( \omega = 2T_s / \Omega \)
(C) \( \Omega = 2\omega / T_s \)
(D) \( \Omega = \omega T_s \)

MCQ 6.4.79

A casual, analog system has a transfer function \( H(s) = \frac{b}{s^2 + a} \). Assuming a sampling time of \( T \) seconds, the poles of the transfer function \( H(z) \) for an equivalent digital system obtained using impulse in variance method are at

(A) \( (e^{aT}, e^{-aT}) \)
(B) \( (i\frac{a}{T}, -i\frac{a}{T}) \)
(C) \( (e^{j\Omega T}, e^{-j\Omega T}) \)
(D) \( (e^{aT/2}, e^{-aT/2}) \)
SOLUTIONS 6.1

SOL 6.1.1 Option (C) is correct. The z-transform is

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{0} \left( - \frac{1}{2} \right)^n z^{-n} \]

\[ = - \sum_{n=0}^{\infty} \left( \frac{1}{2} z^{-1} \right)^n = - \sum_{n=0}^{\infty} \left( \frac{1}{2} z \right)^n \]

\[ = - \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n z^{-n} \]

Let \( n = m \) so,

The above series converges if \( |2z| < 1 \) or \( |z| < \frac{1}{2} \)

\[ X(z) = -\frac{2z}{1-2z} = \frac{2z}{2z-1}, \quad |z| < \frac{1}{2} \]

SOL 6.1.2 Option (A) is correct. We have

\[ x[n] = \left( \frac{1}{2} \right)^n = \left( \frac{1}{2} \right)^0 u[n] + \left( \frac{1}{2} \right)^{n-1} u[-n-1] \]

z-transform is

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]

\[ = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n z^{-n} u[-n-1] + \sum_{n=-\infty}^{\infty} \left( \frac{1}{2} \right)^n z^{-n} u[n] \]

\[ = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n z^{-n} + \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \]

\[ = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n + \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \]

\[ = \frac{1}{1-\frac{1}{2}z} + \frac{z}{1-\frac{1}{2}z} = \frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{2}} \]

Series I converges, if \( \left| \frac{1}{2}z \right| < 1 \) or \( |z| < 2 \)

Series II converges, if \( \left| \frac{1}{2}z \right| < 1 \) or \( |z| > \frac{1}{2} \)

ROC is intersection of both, therefore ROC : \( \frac{1}{2} < |z| < 2 \)

SOL 6.1.3 Option (D) is correct.

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]

\[ = \sum_{n=-\infty}^{\infty} \left( \frac{1}{3} \right)^n z^{-n} u[-n-1] + \sum_{n=-\infty}^{\infty} \left( \frac{1}{3} \right)^n z^{-n} u[n] \]

\[ = \sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^n z^{-n} + \sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^n \]

\[ = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n + \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \]

\[ = \frac{2z}{1-2z} + \frac{1}{1-\frac{1}{2}z} \]

Series I converges, when \( |2z| < 1 \) or \( |z| < \frac{1}{2} \)

Series II converges, when \( \left| \frac{1}{2} \right| < 1 \) or \( |z| > \frac{1}{3} \)
So ROC of \( X(z) \) is intersection of both ROC: \( \frac{1}{3} < |z| < \frac{1}{2} \)

**SOL 6.1.4**

Option (C) is correct.

**z**-transform of \( x[n] \)

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}
\]

\[
= \sum_{n=-\infty}^{\infty} a^n z^{-n} u[n] + \sum_{n=-\infty}^{\infty} \alpha^{-n} z^{-n} u[n]
\]

\[
= \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \sum_{n=0}^{\infty} (\alpha z)^{-n} = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-(\alpha z)^{-1}}
\]

Series I converges, if \( \alpha^{-1} < 1 \) or \( |z| > |\alpha| \)

Series II converges, if \( (\alpha z)^{-1} < 1 \) or \( |z| > \frac{1}{|\alpha|} \)

So ROC is interaction of both

\[
\text{ROC} \ |z| > \max \left( |\alpha|, \frac{1}{|\alpha|} \right)
\]

**SOL 6.1.5**

Option (C) is correct.

\((P \rightarrow 4)\) \( x_1[n] = u[n - 2] \)

\[X_1(z) = \sum_{n=-\infty}^{\infty} u[n - 2]z^{-n} = \sum_{n=-2}^{\infty} z^{-n} = \frac{z^{-2}}{1 - z^{-1}}, \ |z| > 1\]

\((Q \rightarrow 1)\) \( x_2[n] = -u[-n - 3] \)

\[X_2[z] = -\sum_{n=-\infty}^{\infty} u[-n - 3]z^{-n} = \sum_{n=3}^{\infty} z^{-n} \]

\[= \frac{z^3}{1 - z^{-1}} = \frac{1}{z^{-3}(1 - z^{-1})}, \ |z| < 1\]

\((R \rightarrow 3)\) \( x_3[n] = (1)^n u[n + 4] \)

\[X_3(z) = \sum_{n=-\infty}^{\infty} u[n + 4]z^{-n} = \sum_{n=-4}^{\infty} z^{-n} = \frac{z^{-4}}{1 - z^{-1}}, \ |z| > 1\]

\((S \rightarrow 2)\) \( x_4[n] = (1)^n u[-n] \)

\[= \sum_{n=0}^{\infty} u[-n]z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} = \frac{-z^{-1}}{1 - z^{-1}}, \ |z| < 1\]

**SOL 6.1.6**

Option (A) is correct.

\[
X(z) = \sum_{n=-\infty}^{\infty} e^{i\pi z^{-n}} u[n] = \sum_{n=0}^{\infty} (e^{i\pi z^{-1}})^n
\]

\[= \frac{1}{1 - e^{i\pi z^{-1}}}, \ |z| > 1\]

\[= \frac{z}{z - e^{i\pi}} = \frac{z}{z + 1}\]

\[\therefore \ e^{i\pi} = -1\]

**SOL 6.1.7**

Option (D) is correct.

We can write, transfer function

\[
H(z) = \frac{A z^2}{(z - \alpha)(z - \beta)}
\]
\[ H(1) = \frac{A}{(-1)(-2)} = 1 \quad \text{or} \quad A = 2 \]

So,
\[ H(z) = \frac{2z^2}{(z-2)(z-3)} \]
\[ H(z) = \frac{2z}{(z-2)(z-3)} \]
\[ H(z) = -\frac{4z}{(z-2)} + \frac{6z}{(z-3)} \]

From partial fraction

We can see that for \( \text{ROC} : |z| > 3 \), the system is causal and unstable because \( \text{ROC} \) is exterior of the circle passing through outermost pole and does not include unit circle.

so,
\[ h[n] = [(−4)2^n + (6)3^n]u[n], \quad |z| > 3 \quad \text{(P \rightarrow 2)} \]

For \( \text{ROC} : 2 < |z| < 3 \). The sequence corresponding to pole at \( z = 2 \) corresponds to right-sided sequence while the sequence corresponds to pole at \( z = 3 \) corresponds to left sided sequence
\[ h[n] = (−4)2^n u[n] + (−6)3^n u[−n − 1] \quad \text{(Q \rightarrow 4)} \]

For \( \text{ROC} : |z| < 2 \), \( \text{ROC} \) is interior to circle passing through inner most pole, hence the system is non causal.

\[ h[n] = (4)2^n u[−n − 1] + (−6)3^n u[−n − 1] \quad \text{(R \rightarrow 3)} \]

For the response
\[ h[n] = 4(2)^n u[−n − 1] + (−6)3^n u[n] \]

\( \text{ROC} : |z| < 2 \) and \( |z| > 3 \) which does not exist \( \text{(S \rightarrow 1)} \)

SOL 6.1.8

Option (A) is correct.

\[ X(z) = e^z + e^{2z} \]
\[ X(z) = (1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + .......) + (1 + \frac{1}{z} + \frac{1}{2!z^2} + .......) \]
\[ = (1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + .......) + (1 + z^{-1} + \frac{z^{-2}}{2!} + .......) \]

Taking inverse z-transform
\[ x[n] = \delta[n] + \frac{1}{n!} \]

SOL 6.1.9

Option (A) is correct.

\[ X(z) = \frac{z^2 + 5z}{z^2 - 2z - 3} = \frac{z(z + 5)}{(z-3)(z+1)} \]
\[ X(z) = \frac{z + 5}{(z-3)(z+1)} = \frac{2}{z-3} - \frac{1}{z+1} \]

By partial fraction

Thus
\[ X(z) = \frac{2z}{z-3} - \frac{z}{z+1} \]

Poles are at \( z = 3 \) and \( z = -1 \)

ROC : \( |z| < 1 \), which is not exterior of circle outside the outermost pole \( z = 3 \). So, \( x[n] \) is anticausal given as
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\[ x[n] = [-2(3)^n + (-1)^n]u[-n - 1] \]

**SOL 6.1.10**

Option (A) is correct.

\[ X(z) = \frac{2z}{z-3} - \frac{z}{z+1} \]

If \(|z| > 3\), ROC is exterior of a circle outside the outer most pole, \(x[n]\) is causal.

**SOL 6.1.11**

Option (C) is correct.

\[ X(z) = \frac{2z}{z-3} - \frac{z}{z+1} \]

If ROC is \(1 < |z| < 3\), \(x[n]\) is two sided with anticausal part \(\frac{2z}{z-3}, |z| < 3\) and causal part \(\frac{-z}{z+1}, |z| > 1\)

\[ x[n] = [2(3)^n - (-1)^n]u[n] \]

**SOL 6.1.12**

Option (D) is correct.

\[ X_1(z) = \sum_{n=-\infty}^{\infty} (0.7)^n z^{-n} u[n-1] = \sum_{n=1}^{\infty} (0.7z^{-1})^n \]
\[ = \frac{0.7z^{-1}}{1-0.7z^{-1}} \]

ROC : \(|0.7z^{-1}| < 1 \) or \(|z| > 0.7\)

\[ X_2(z) = \sum_{n=-\infty}^{\infty} (-0.4)^n z^{-n} u[-n-2] = \sum_{n=-2}^{\infty} (-0.4)^n z^{-n} \]
\[ = \sum_{m=-2}^{\infty} (-0.4)^{-m} z^m \]
\[ = \sum_{m=-2}^{\infty} [(-0.4)^{-1}]^m z^m = \sum_{m=-2}^{\infty} \frac{(-0.4)^{-1}z}{1+(0.4)^{-1}z} \]

ROC : \(|(0.4)^{-1}z| < 1 \) or \(|z| < 0.4\)

The ROC of \(z\)-transform of \(x[n]\) is intersection of both which does not exist.

**SOL 6.1.13**

Option (D) is correct.
If \( x[n] \xrightarrow{Z} X(z) \) from time shifting property
\[ x[n - n_0] \xrightarrow{Z} z^{-n_0}X(z) \]
So
\[ z(x[n - 4]) = z^{-4}X(z) = \frac{1}{8z^2 - 2z^4 - z^5} \]

**SOL 6.1.14**

Option (A) is correct.
We can see that
\[ X_1(z) = z^2X_2(z) = z^2X_3(z) \]
or
\[ z^{-2}X_1(z) = z^{-1}X_2(z) = X_3(z) \]
So
\[ x[n - 2] = x[n - 1] = x[n] \]

**SOL 6.1.15**

Option (C) is correct.
\( x[n] \) can be written in terms of unit sequence as
\[ x[n] = u[n] - u[n - k] \]
so
\[ X(z) = \frac{z}{z - 1} - \frac{z^k}{z - 1} = \frac{1 - z^k}{1 - z^{-1}} \]

**SOL 6.1.16**

Option (C) is correct.
For positive shift
If,
\[ x[n] \xrightarrow{Z} X(z) \]
then,
\[ x[n - n_0] \xrightarrow{Z} z^{-n_0}X(z), \quad n_0 \geq 0 \]
So
\[ x[n - 1] \xrightarrow{Z} z^{-1}(X(z) - \sum_{m=0}^{n_0 - 1} x[n]z^{-m}), \quad n_0 > 0 \]
\[ x[n + 1] \xrightarrow{Z} z(X(z) - x[0]) \]
We know that \( x[n] = u[n] \) so \( x[0] = 1 \) and
\[ x[n + 1] \xrightarrow{Z} z(X(z) - 1) = z\left(\frac{z}{z - 1} - 1\right) = \frac{z}{z - 1} \]

**SOL 6.1.17**

Option (B) is correct.
Even part of \( x[n] \), \( x_e[n] = \frac{1}{2}(x[n] + x[-n]) \)
z- transform of \( x_e[n] \), \( X_e(z) = \frac{1}{2}X(z) + X\left(\frac{1}{z}\right) \)
\[ = \frac{1}{2}\left(\frac{z}{z - 0.4}\right) + \frac{1}{2}\left(\frac{1/4}{1/4 - 0.4}\right) \]

Region of convergence for I series is \( |z| > 0.4 \) and for II series it is \( |z| < 2.5 \).
Therefore, \( X_e(z) \) has ROC \( 0.4 < |z| < 2.5 \).

**SOL 6.1.18**

Option (B) is correct.
(P \( \to 4 \)) \( y[n] = n(-1)^nu[n] \)
We know that
\[ (-1)^nu[n] \xrightarrow{Z} \frac{1}{1+z^{-1}}, \quad |z| > 1 \]
If
\[ x[n] \xrightarrow{Z} X(z) \]
then,
\[ nx[n] \xrightarrow{Z} -z\frac{dX(z)}{dz} \quad (z\text{-domain differentiation}) \]
so, \[ n(-1)^nu[n] \rightarrow z^{-1} \frac{d}{dz} \left[ \frac{1}{1+z^{-1}} \right], \quad \text{ROC}: |z| > 1 \]

\[ Y(z) = \frac{-z^{-1}}{(1+z^{-1})^2}, \quad \text{ROC}: |z| > 1 \]

(Q \rightarrow 3) \quad y[n] = -nu[-n-1]

We know that,

\[ u[-n-1] \rightarrow z^{-1} \frac{-1}{(1-z^{-1})}, \quad \text{ROC}: |z| < 1 \]

Again applying z-domain differentiation property

\[-nu[-n-1] \rightarrow z^{-1} \frac{d}{dz} \left[ \frac{-1}{1-z^{-1}} \right], \quad \text{ROC}: |z| < 1 \]

\[ Y(z) = \frac{z^{-1}}{(1-z^{-1})^2}, \quad \text{ROC}: |z| < 1 \]

(R \rightarrow 2) \quad y[n] = (-1)^nu[n]

\[ Y(z) = \sum_{n=0}^{\infty} (-1)^nu[n] = \sum_{n=0}^{\infty} (-z^{-1})^n = \frac{1}{1+z^{-1}}, \quad \text{ROC}: |z| > 1 \]

(S \rightarrow 1) \quad y[n] = nu[n]

We know that \[ u[n] \rightarrow \frac{1}{1-z^{-1}}, \quad \text{ROC}: |z| > 1 \]

so,

\[ nu[n] \rightarrow z^{-1} \frac{d}{dz} \left[ \frac{1}{1-z^{-1}} \right], \quad \text{ROC}: |z| > 1 \]

\[ Y(z) = \frac{z^{-1}}{(1-z^{-1})^2}, \quad \text{ROC}: |z| > 1 \]

SOL 6.1.19

Option (A) is correct.

It's difficult to obtain the z-transform of \( x[n] \) directly due to the term \( 1/n \).

Let \[ y[n] = nx[n] = (-2)^nu[-n-1] \]

So z-transform of \( y[n] \)

\[ Y(z) = \frac{-z}{z + \frac{1}{z}}, \quad \text{ROC}: |z| < \frac{1}{2} \]

Since \[ y[n] = nx[n] \]

so,

\[ Y(z) = -z \frac{dX(z)}{dz} \quad \text{(Differentiation in z-domain)} \]

\[ -z \frac{dX(z)}{dz} = \frac{-z}{z + \frac{1}{z}} \]

\[ \frac{dX(z)}{dz} = \frac{1}{z + \frac{1}{z}} \]

or

\[ X(z) = \log \left( z + \frac{1}{2} \right), \quad \text{ROC}: |z| < \frac{1}{2} \]

SOL 6.1.20

Option (A) is correct.

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad \text{ROC}: \Re(z) \]

\[ Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} a^n x[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n] \left( \frac{z}{a} \right)^{-n} = X \left( \frac{z}{a} \right), \quad \text{ROC}: \Re(a) \]

SOL 6.1.21

Option (B) is correct.

Using time shifting property of z-transform

If, \[ x[n] \rightarrow z^{-1} \rightarrow X(z), \quad \text{ROC}: \Re(z) \]

\[ x[n] \rightarrow z^{-1} \rightarrow X(z), \quad \text{ROC}: \Re(z) \]
then, \[ x[n - n_0] \overset{Z}{\rightarrow} z^{-n_0}X(z) \]
with same ROC except the possible deletion or addition of \( z = 0 \) or \( z = \infty \).
So, ROC for \( x[n - 2] \) is \( R_x \) (\( S_3, R_3 \))
Similarly for \( x[n + 2] \), ROC : \( R_x \) (\( S_2, R_3 \))
Using time-reversal property of \( z \)-transform
If, \[ x[n] \overset{Z}{\rightarrow} X(z) \text{, ROC : } R_x \]
then, \[ x[-n] \overset{Z}{\rightarrow} X\left(\frac{1}{Z}\right) \text{, ROC : } \frac{1}{R_x} \]
For \( S_3 \),
\[ x[-n] \overset{Z}{\rightarrow} X\left(\frac{1}{Z}\right), \]
Because \( z \) is replaced by \( 1/z \), so ROC would be \( |z| < \frac{1}{\alpha} \) (\( S_3, R_3 \))
\( S_4 : (−1)^n x[n] \)
Using the property of scaling in \( z \)-domain, we have
If, \[ x[n] \overset{Z}{\rightarrow} X(z) \text{, ROC : } R_x \]
then, \[ \alpha^n x[n] \overset{Z}{\rightarrow} X\left(\frac{Z}{\alpha}\right) \]
\( z \) is replaced by \( z/\alpha \) so ROC will be \( R_x \) (\( S_3, R_3 \))
Here \( (−1)^n x[n] \overset{Z}{\rightarrow} X\left(\frac{Z}{\alpha}\right), \quad \alpha = 1 \)
so,\[ \text{ROC : } |z| > \frac{1}{\alpha} \] (\( S_4, R_4 \))
Option (D) is correct.

**Time scaling property** :
If, \[ x[n] \overset{Z}{\rightarrow} X(z) \]
then, \[ x[n/2] \overset{Z}{\rightarrow} X(z^2) \] (P → 2)

**Time shifting property** :
If, \[ x[n] \overset{Z}{\rightarrow} X(z) \]
then, \[ x[n - 2]u[n - 2] \overset{Z}{\rightarrow} z^{-2}X(z) \] (Q → 1)
For \( x[n + 2]u[n] \) we can not apply time shifting property directly.
Let, \[ y[n] = x[n + 2]u[n] \]
\[ = \alpha^{n+2}u[n + 2]u[n] = \alpha^{n+2}u[n] \]
so, \[ Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=0}^{\infty} \alpha^{n+2}z^{-n} \]
\[ = \alpha^2 \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \alpha^2 X(z) \] (R → 4)
Let, \[ g[n] = \beta^{2n}x[n] \]
\[ G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n} = \sum_{n=-\infty}^{\infty} \beta^{2n} \alpha^n z^{-n}u[n] \]
\[ = \sum_{n=0}^{\infty} \alpha^n \left(\frac{z}{\beta^2}\right)^n = X\left(\frac{Z}{\beta^2}\right) \] (S → 3)

Option (C) is correct.
Let, \[ y[n] = x[2n] \]
\[ Y(z) = \sum_{n=-\infty}^{\infty} x[2n]z^{-n} \]
\[ = \sum_{k=-\infty}^{\infty} x[k]z^{-k/2} \]
Put \( 2n = k \) or \( n = \frac{k}{2} \), \( k \) is even

Since \( k \) is even, so we can write
SOL 6.1.24

Option (A) is correct.
From the accumulation property we know that
If, 
\[ x[n] \sim \mathcal{Z} \rightarrow X(z) \]
then,
\[ \sum_{k=-\infty}^{n} x[k] \sim \mathcal{Z} \rightarrow \frac{z}{(z-1)} X(z) \]

Here,
\[ y[n] = \sum_{k=0}^{n} x[k] \]
\[ Y(z) = \frac{z}{(z-1)} X(z) = \frac{4z^2}{(z-1)(8z^2 - 2z - 1)} \]

SOL 6.1.25

Option (B) is correct.
By taking \( z \)-transform of both the sequences
\[ X(z) = (-1 + 2z^{-1} + 0 + 3z^{-3}) \]
\[ H(z) = 2z^2 + 3 \]
Convolution of sequences \( x[n] \) and \( h[n] \) is given as
\[ y[n] = x[n] * h[n] \]
Applying convolution property of \( z \)-transform, we have
\[ Y(z) = X(z) H(z) \]
\[ = (-1 + 2z^{-1} + 3z^{-3})(2z^2 + 3) = -2z^2 + 4z - 3 + 12z^{-1} + 9z^{-3} \]
or,
\[ y[n] = \{-2, 4, -3, 12, 0, 9\} \]

SOL 6.1.26

Option (C) is correct.
The \( z \)-transform of signal \( x^*[n] \) is given as follows
\[ \mathcal{Z}\{x^*[n]\} = \sum_{n=-\infty}^{\infty} x^*[n] z^{-n} = \sum_{n=-\infty}^{\infty} [x[n](z^*)^{-n}] \]  
\[ = \frac{\sum_{n=-\infty}^{\infty} x[n] z^{-n}}{z^*} \]  
Let \( z \)-transform of \( x[n] \) is \( X(z) \)
\[ X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]
Taking complex conjugate on both sides of above equation
\[ X^*(z) = \sum_{n=-\infty}^{\infty} [x[n] z^{-n}]^* \]
Replacing \( z \rightarrow z^* \), we will get
\[ X^*(z^*) = \sum_{n=-\infty}^{\infty} [x[n] (z^*)^{-n}]^* \]  
Comparing equation (1) and (2)
\[ \mathcal{Z}\{x^*[n]\} = X^*(z^*) \]

SOL 6.1.27

Option (B) is correct.
By taking \( z \)-transform on both sides of given difference equation
\[ Y(z) - \frac{1}{2}z^{-1} Y(z) + y[-1]z = X(z) \]
Let impulse response is \( H(z) \), so the impulse input is \( X(z) = 1 \)
\[ H(z) - \frac{1}{2}z^{-1}[H(z) + 3z] = 1 \]
SOL 6.1.28
Option (B) is correct.

\[ h[n] = (2)^n u[n] \]

Taking z-transform

\[ H(z) = \frac{z}{z - 2} = \frac{Y(z)}{X(z)} \]

so,

\[(z - 2)Y(z) = zX(z)\]

or,

\[(1 - 2z^{-1})Y(z) = X(z)\]

Taking inverse z-transform

\[ y[n] - 2y[n - 1] = x[n] \]

SOL 6.1.29
Option (B) is correct.

\[ h[n] = \delta[n] - \left(\frac{-1}{2}\right)^n u[n] \]

z-transform of \( h[n] \)

\[ H(z) = 1 - \frac{z}{z + \frac{1}{2}} = \frac{\frac{1}{2}}{z + \frac{1}{2}} = \frac{Y(z)}{X(z)} \]

\[(z + \frac{1}{2})Y(z) = \frac{1}{2}X(z)\]

\[(1 + \frac{1}{2}z^{-1})Y(z) = \frac{1}{2}z^{-1}X(z)\]

Taking inverse z-transform

\[ y[n] + \frac{1}{2}y[n - 1] = \frac{1}{2}x[n - 1] \]

\[ y[n] + 0.5y[n - 1] = 0.5x[n - 1] \]

SOL 6.1.30
Option (C) is correct.

We have

\[ y[n] - 0.4y[n - 1] = (0.4)^n u[n] \]

Zero state response refers to the response of system with zero initial condition.

So, by taking z-transform

\[ Y(z) - 0.4z^{-1}Y(z) = \frac{Z}{z - 0.4} \]

\[ Y(z) = \frac{Z}{(z - 0.4)^2} \]

Taking inverse z-transform

\[ y[n] = (n + 1)(0.4)^n u[n] \]

SOL 6.1.31
Option (B) is correct.

Zero state response refers to response of the system with zero initial conditions.

Taking z-transform

\[ Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) \]

\[ Y(z) = \left(\frac{Z}{Z - 0.5}\right)X(z) \]

For an input

\[ x[n] = u[n], \quad X(z) = \frac{Z}{Z - 1} \]
so,

\[ Y(z) = \frac{z}{(z-0.5)(z-1)} = \frac{z^2}{(z-1)(z-0.5)} \]

\[ Y(z) = \frac{z}{(z-1)(z-0.5)} \]

\[ = \frac{2}{z-1} - \frac{1}{z-0.5} \]

By partial fraction

Thus

\[ Y(z) = \frac{2z}{z-1} - \frac{z}{z-0.5} \]

Taking inverse z-transform

\[ y[n] = 2u[n] - (0.5)^n u[n] \]

SOL 6.1.32

Option (C) is correct.

Input,

\[ x[n] = 2\delta[n] + \delta[n + 1] \]

By taking z-transform

\[ X(z) = 2 + z \]

\[ Y(z) = H(z), \quad Y(z) \text{ is z-transform of output } y[n] \]

\[ Y(z) = H(z)X(z) \]

\[ = \frac{2z(z-1)}{(z+2)^2}(z+2) \]

\[ = \frac{2z(z-1)}{(z+2)} = 2z - \frac{6z}{z+2} \]

Taking inverse z-transform

\[ y[n] = 2\delta[n + 1] - 6(-2)^n u[n] \]

SOL 6.1.33

Option (B) is correct.

Poles of the system function are at \( z = \pm j \) ROC is shown in the figure.

**Causality :**

We know that a discrete time LTI system with transfer function \( H(z) \) is causal if and only if ROC is the exterior of a circle outside the outer most pole.

For the given system ROC is exterior to the circle outside the outer most pole \( (z = \pm j) \). The system is causal.

**Stability :**

A discrete time LTI system is stable if and only if ROC of its transfer function \( H(z) \) includes the unit circle \( |z| = 1 \).

The given system is unstable because ROC does not include the unit circle.

**Impulse Response :**

\[ H(z) = \frac{z}{z^2 + 1} \]

We know that
\[
\sin(\Omega_0 n) u[n] \xrightarrow{z} \frac{z \sin \Omega_0}{z^2 - 2z \cos \Omega_0 + 1}, \quad |z| > 1
\]

Here
\[
z^2 + 1 = z^2 - 2z \cos \Omega_0 + 1
\]
So
\[
2z \cos \Omega_0 = 0 \quad \text{or} \quad \Omega_0 = \frac{\pi}{2}
\]

Taking the inverse Laplace transform of \( H(z) \)
\[
h[n] = \sin\left(\frac{\pi}{2} n\right) u[n]
\]

**SOL 6.1.34**
Option (D) is correct.  
Statement (A), (B) and (C) are true.

**SOL 6.1.35**
Option (D) is correct.  
First we obtain transfer function (z-transform of \( h[n] \)) for all the systems and then check for stability.

(A)  
\[
H(z) = \frac{z}{z^2 - \frac{\pi}{2}}
\]
Stable because all poles lies inside unit circle.

(B)  
\[
h[n] = \frac{1}{3} \delta[n]
\]
\[
\sum |h[n]| = \frac{1}{3}
\]
( absolutely summable)
Thus this is also stable.

(C)  
\[
h[n] = \delta[n] - \frac{1}{3} \delta[n-1] u[n]
\]
\[
H(z) = 1 - \frac{z}{z + \frac{2}{3}}
\]
Pole is inside the unit circle, so the system is stable.

(D)  
\[
h[n] = [(2)^n - (3)^n] u[n]
\]
\[
H(z) = \frac{z}{z - 2} - \frac{z}{z - 3}
\]
Poles are outside the unit circle, so it is unstable.

**SOL 6.1.36**
Option (B) is correct.  
By taking z-transform
\[
(1 + 3z^{-1} + 2z^{-2}) Y(z) = (2 + 3z^{-1}) X(z)
\]
So, transfer function
\[
H(z) = \frac{Y(z)}{X(z)} = \frac{(2 + 3z^{-1})}{(1 + 3z^{-1} + 2z^{-2})} = \frac{2z^2 + 3z}{z^2 + 3z + 2}
\]
or
\[
H(z) = \frac{2z + 3}{z^2 + 3z + 2} = \frac{1}{z + 2} + \frac{1}{z + 1}
\]
By partial fraction
Thus
\[
H(z) = \frac{z}{z + 2} + \frac{z}{z + 1}
\]
Both the poles lie outside the unit circle, so the system is unstable.

**SOL 6.1.37**
Option (B) is correct.  
\[
y[n] = x[n] + y[n-1]
\]
Put \( x[n] = \delta[n] \) to obtain impulse response \( h[n] \)
\[
h[n] = \delta[n] + h[n-1]
\]
For \( n = 0 \),
\[
h[0] = \delta[0] + h[-1]
\]
\[
h[0] = 1
\]
\[
\text{(} h[-1] = 0, \text{for causal system)}
\]
\[
n = 1,
\]
\[
h[1] = \delta[1] + h[0]
\]
Sample Chapter of **Signals and Systems** (Vol-7, GATE Study Package)

The Z-Transform

h[1] = 1


h[2] = 1

In general form

h[n] = u[n]

Thus, statement 1 is true.

Let

\[ x[n] \xrightarrow{Z} X(z) \]

\[ X(z) = \frac{z}{z - 0.5} \]

\[ h[n] \xrightarrow{Z} H(z) \]

\[ H(z) = \frac{z}{z - 1} \]

Output

\[ Y(z) = H(z) X(z) \]

\[ = \left( \frac{z}{z - 1} \right) \left( \frac{z}{z - 0.5} \right) = \frac{2z}{z - 1} - \frac{z}{z - 0.5} \]

By partial fraction

Inverse z-transform

\[ y[n] = 2u[n] - (0.5)^n u[n] \]

Statement 3 is also true.

\[ H(z) = \frac{z}{z - 1} \]

System pole lies at unit circle |z| = 1, so the system is not BIBO stable.

**SOL 6.1.38**

Option (C) is correct.

(P → 3) ROC is exterior to the circle passing through outer most pole at z = 1.2, so it is causal. ROC does not include unit circle, therefore it is unstable.

(Q → 1) ROC is not exterior to the circle passing through outer most pole at z = 1.2, so it is non causal. But ROC includes unit circle, so it is stable.

(R → 2), ROC is not exterior to circle passing through outermost pole z = 0.8 , so it is not causal, ROC does not include the unit circle, so it is unstable also.

(S → 4), ROC contains unit circle and is exterior to circle passing through outermost pole, so it is both causal and stable.

**SOL 6.1.39**

Option (D) is correct.

\[ H(z) = \frac{P(z - 0.9)}{z - 0.9 + Pz} \]

\[ = \frac{P(z - 0.9)}{(1 + P)z - 0.9} = \frac{P}{1 + P} \left( \frac{z - 0.9}{z - \frac{0.9}{1 + P}} \right) \]

Pole at

\[ z = \frac{0.9}{1 + P} \]

For stability pole lies inside the unit circle, so

\[ |z| < 1 \]

or

\[ \left| \frac{0.9}{1 + P} \right| < 1 \]

\[ 0.9 < |1 + P| \]

\[ P > -0.1 \text{ or } P < -1.9 \]

**SOL 6.1.40**

Option (A) is correct.

For a system to be causal and stable, \( H(z) \) must not have any pole outside the unit circle \(|z| = 1\).

\[ S_1 : \quad H(z) = \frac{z - \frac{1}{2}}{z^2 + \frac{1}{2}z - \frac{1}{16}} = \frac{z - \frac{1}{2}}{(z - \frac{1}{4})(z + \frac{3}{4})} \]
Poles are at \( z = 1/4 \) and \( z = -3/4 \), so it is causal.

**S2:**
\[
H(z) = \frac{z + 1}{(z + \frac{1}{4})(1 - \frac{1}{4}z^{-2})}
\]
one pole is at \( z = -4/3 \), which is outside the unit circle, so it is not causal.

**S3:** one pole is at \( z = \infty \), so it is also non-causal.

**SOL 6.1.41**
Option (D) is correct.

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n - k]z^{-n} = z^k, \quad z \neq 0
\]

**ROC:** We can find that \( X(z) \) converges for all values of \( z \) except \( z = 0 \), because at \( z = 0 \) \( X(z) \to \infty \).

**SOL 6.1.42**
Option (D) is correct.

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n + k]z^{-n} = z^k, \quad \text{all } z
\]

**ROC:** We can see that above summation converges for all values of \( z \).

**SOL 6.1.43**
Option (A) is correct.

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} u[n]z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}
\]

**ROC:** Summation I converges if \( |z| > 1 \), because when \( |z| < 1 \), then \( \sum z^{-n} \to \infty \).

**SOL 6.1.44**
Option (D) is correct.

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}
\]

\[
= \sum_{n=-\infty}^{\infty} \left( \frac{1}{4} \right)^n (u[n] - u[n - 5])z^{-n}
\]

\[
= \sum_{n=0}^{4} \left( \frac{1}{4}z^{-1} \right)^n u[n] - u[n - 5] = 1, \quad \text{for } 0 \leq n \leq 4
\]

\[
= \frac{1 - (\frac{1}{4}z^{-1})^5}{1 - (\frac{1}{4}z^{-1})} = \frac{z^5 - (0.25)^5}{z^5(z - 0.5)}, \quad \text{all } z
\]

**ROC:** Summation I converges for all values of \( z \) because \( n \) has only four value.

**SOL 6.1.45**
Option (D) is correct.

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}
\]

\[
= \sum_{n=-\infty}^{\infty} \left( \frac{1}{4} \right)^n u[-n]z^{-n}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{4}z^{-1} \right)^n = \sum_{n=0}^{\infty} (4z)^{-n}
\]

\[
= \sum_{n=0}^{\infty} (4z)^{-n} = \frac{1}{1-4z}, \quad |z| < \frac{1}{4}
\]

**ROC:** Summation I converges if \( |4z| < 1 \) or \( |z| < \frac{1}{4} \).

**SOL 6.1.46**
Option (B) is correct.

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} 3^n u[-n - 1]z^{-n}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{3}z^{-1} \right)^n u[-n - 1] = 1, \quad n \leq -1
\]
\[ z = \frac{1}{z} = \frac{2}{3} \quad \text{or} \quad |z| < 3 \]

**ROC**: Summation I converges when \(|\frac{1}{z}| < 1\) or \(|z| < 3\)

**SOL 6.1.47**

Option (B) is correct.

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{2}{3}\right)^n z^{-n} \]
\[ = \sum_{n=-\infty}^{-1} \left(\frac{2}{3}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n z^{-n} \]

In first summation taking \(n = -m\),

\[ X(z) = \sum_{m=1}^{\infty} \left(\frac{2}{3}\right)^m z^{-m} + \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n z^{-n} \]
\[ = \sum_{m=1}^{\infty} \left(\frac{2}{3}\right)^m + \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n z^{-n} \]
\[ = \frac{\frac{2}{3}}{1 - \frac{1}{3}z^{-1}} + \frac{1}{1 - \frac{1}{3}z^{-1}} \]
\[ = \frac{1}{1 - \frac{1}{3}z^{-1}} \]

**ROC**: Summation I converges if \(|\frac{1}{z}| < 1\) or \(|z| < \frac{3}{2}\) and summation II converges if \(|\frac{1}{3}z^{-1}| < 1\) or \(|z| > \frac{3}{2}\). ROC of \(X(z)\) would be intersection of both, that is \(\frac{3}{2} < |z| < \frac{3}{2}\)

**SOL 6.1.48**

Option (B) is correct.

\[ x[n] = \cos\left(\frac{\pi}{3} n\right) u[n] = \frac{e^{j(\pi)n} + e^{-j(\pi)n}}{2} u[n] \]
\[ = \frac{1}{2} e^{j(\pi)n} u[n] + \frac{1}{2} e^{-j(\pi)n} u[n] \]

\[ X(z) = \sum_{n=0}^{\infty} a^n u[n] z^{-n} \rightarrow \frac{1}{1 - az^{-1}}, \quad |z| > |a| \]
\[ = \frac{1}{2} \left[ 1 - \frac{1}{2} z^{-1} \right] \left[ 1 - \frac{1}{2} e^{j\pi} z^{-1} \right] \]
\[ = \frac{1}{2} \left[ 1 - \frac{2 - \frac{1}{2} z^{-1} \left(e^{j\pi} + e^{-j\pi}\right) + \frac{z^{-2}}{2} \right], \quad |z| > 1 \]

**ROC**: First term in \(X(z)\) converges for \(|z| > |e^{j\pi}| \Rightarrow |z| > 1\). Similarly II term also converges for \(|z| > |e^{-j\pi}| \Rightarrow |z| > 1\), so ROC would be simply \(|z| > 1\).

**SOL 6.1.49**

Option (B) is correct.

\[ x[n] = 3\delta[n + 5] + 6\delta[n] + \delta[n - 1] - 4\delta[n - 2] \]
\[ X(z) = 3z^5 + 6z^4 + \delta[n - 1] + 4z^{-2}, \quad 0 < |z| < \infty \]
\[ \delta[n \pm n_0] \rightarrow z^{\pm n_0} \]

**ROC**: \(X(z)\) is finite over entire \(z\) plane except \(z = 0\) and \(z = \infty\) because when \(z = 0\) negative power of \(z\) becomes infinite and when \(z \rightarrow \infty\) the positive powers of \(z\) tends to becomes infinite.

**SOL 6.1.50**

Option (D) is correct.

\[ x[n] = 2\delta[n + 2] + 4\delta[n + 1] + 5\delta[n] + 7\delta[n - 1] + \delta[n - 3] \]
\[ X(z) = 2z^2 + 4z + 5z^{-1} + 7z^{-2} + \delta[n \pm n_0] \rightarrow z^{\pm n_0} \]
\[ \delta[n \pm n_0] \rightarrow z^{\pm n_0} \]
SOL 6.1.51
Option (B) is correct.

\[ x[n] = \delta[n] - \delta[n - 2] + \delta[n - 4] - \delta[n - 5] \]

\[ X(z) = 1 - z^{-2} + z^{-4} - z^{-5}, \quad z \neq 0 \]

\[ \delta[n] \rightarrow z^{-n} \]

**ROC :** \( X(z) \) has only negative powers of \( z \), therefore transform \( X(z) \) does not converge for \( z = 0 \).

SOL 6.1.52
Option (A) is correct.

Using partial fraction expansion, \( X(z) \) can be simplified as

\[ X(z) = \frac{z^2 - 3z}{z^2 + \frac{1}{2}z - 1} = \frac{1 - 3z^{-1}}{1 + \frac{1}{2}z^{-1} - z^{-2}} \]

\[ = \frac{2}{1 + 2z^{-1}} - \frac{1}{1 - \frac{1}{2}z^{-1}} \]

Poles are at \( z = -2 \) and \( z = \frac{1}{2} \). We obtain the inverse z-transform using relationship between the location of poles and region of convergence as shown in the figure.

ROC : \( \frac{1}{2} < |z| < 2 \) has a radius less than the pole at \( z = -2 \) therefore the \( I \) term of \( X(z) \) corresponds to a left sided signal

\[ \frac{2}{1 + 2z^{-1}} \cdot z^{-1} \cdot -2(2)^n u[-n - 1] \quad \text{(left-sided signal)} \]

While, the ROC has a greater radius than the pole at \( z = \frac{1}{2} \), so the second term of \( X(z) \) corresponds to a right sided sequence.

\[ \frac{1}{1 - \frac{1}{2}z^{-1}} \cdot z^{-1} \cdot \frac{1}{2^n} u[n] \quad \text{(right-sided signal)} \]

So, the inverse z-transform of \( X(z) \) is

\[ x[n] = -2(2)^n u[-n - 1] - \frac{1}{2^n} u[n] \]

SOL 6.1.53
Option (A) is correct.

Using partial fraction expansion \( X(z) \) can be simplified as follows

\[ X(z) = \frac{3z^2 - \frac{1}{2}z}{z^2 - 16} = \frac{3 - \frac{1}{2}z^{-1}}{1 - 16z^{-1}} \]

\[ = \frac{49}{32} \frac{1}{1 + 4z^{-1}} + \frac{47}{27} \frac{1}{1 - 4z^{-1}} \]

Poles are at \( z = -4 \) and \( z = 4 \). Location of poles and ROC is shown in the figure.
ROC : |z| > 4 has a radius greater than both the poles at z = -4 and z = 4, therefore both the terms in X(z) corresponds to right sided sequences. Taking inverse z-transform we have

\[ x[n] = \left[ \frac{49}{32}(-4)^n + \frac{47}{32}4^n \right] u[n] \]

Option (C) is correct.

Using partial fraction expansion X(z) can be simplified as

\[ X(z) = \frac{2z^2 - 2z - 2z^3}{z^2 - 1} \]
\[ = \frac{2 - 2z^{-1} - 2z^{-2}}{1 - z^{-2}} \]
\[ = \left[ 2 + \frac{1}{1+z^{-1}} + \frac{-1}{1-z^{-1}} \right] z^2 \]

Poles are at z = -1 and z = 1. Location of poles and ROC is shown in the following figure

ROC : |z| > 1 has radius grater than both the poles at z = -1 and z = 1, therefore both the terms in X(z) corresponds to right sided sequences.

\[ \frac{1}{1+z^{-1}} \rightarrow z^{-1}(-1)^n u[n] \] (right-sided)
\[ \frac{1}{1-z^{-1}} \rightarrow u[n] \] (right-sided)

Now, using time shifting property the complete inverse z-transform of X(z) is

\[ x[n] = 2\delta[n + 2] + ((-1)^n - 1) u[n + 2] \]
Option (A) is correct.

We have, \( X(z) = 1 + 2z^{-6} + 4z^{-8} \), \( |z| > 0 \)

Taking inverse z-transform we get
\[
x[n] = \delta[n] + 2\delta[n - 6] + 4\delta[n - 8]
\]
\[ z^{-n_0} \xrightarrow{Z^{-1}} \delta[n - n_0] \]

Option (B) is correct.

Since \( x[n] \) is right sided, \( x[n] = \sum_{k=0}^{10} \frac{1}{k} \delta[n - k] \)

Taking inverse z-transform we get
\[
x[n] = \delta[n] + 3\delta[n - 1] + 3\delta[n - 2] + \delta[n - 3]
\]
\[ z^{-n_0} \xrightarrow{Z^{-1}} \delta[n - n_0] \]

Option (D) is correct.

We have, \( X(z) = (1 + z^{-1})^3 = 1 + 3z^{-1} + 3z^{-2} + z^{-3} \), \( |z| > 0 \)

Since \( x[n] \) is right sided signal, taking inverse z-transform we have
\[
x[n] = \delta[n] + 3\delta[n - 1] + 3\delta[n - 2] + \delta[n - 3]
\]
\[ z^{-n_0} \xrightarrow{Z^{-1}} \delta[n - n_0] \]

Option (A) is correct.

We have, \( X(z) = z^6 + z^2 + 3 + 2z^{-3} + z^{-4} \), \( |z| > 0 \)

Taking inverse z-transform we get
\[
x[n] = \delta[n + 6] + \delta[n + 2] + 3\delta[n] + 2\delta[n - 3] + \delta[n - 4]
\]

Option (A) is correct.

\[
X(z) = \frac{1}{1 - \frac{1}{4}z^{-2}}, \quad |z| > \frac{1}{2}
\]

The power series expansion of \( X(z) \), with \( |z| > \frac{1}{2} \) or \( |\frac{1}{2}z^{-2}| < 1 \) is written as
\[
X(z) = 1 + \left(\frac{1}{4}\right)z^{-2} + \left(\frac{1}{4}\right)^2 z^{-4} + \left(\frac{1}{4}\right)^3 z^{-6} + \cdots
\]

Series converges for \( \left|\frac{1}{2}z^{-2}\right| < 1 \) or \( |z| > \frac{1}{2} \). Taking inverse z-transform we get
\[
x[n] = \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \delta[n - 2k]
\]
\[ z^{-2k} \xrightarrow{Z^{-1}} \delta[n - 2k] \]

\[ n \text{ even and } n \geq 0 \]
\[ 0, \quad n \text{ odd} \]

\[ n \text{ even and } n \geq 0 \]
\[ 0, \quad n \text{ odd} \]

Option (C) is correct.

\[
X(z) = \frac{1}{1 - \frac{1}{4}z^{-2}}, \quad |z| < \frac{1}{2}
\]

Since ROC is left sided so power series expansion of \( X(z) \) will have positive powers of \( z \), we can simplify above expression for positive powers of \( z \) as
\[
X(z) = -4z^2 \left[1 + (2z)^2 + (2z)^4 + (2z)^6 + \cdots\right]
\]
\[
X(z) = -4z^2 \sum_{k=0}^{\infty} (2z)^{2k} = -\sum_{k=0}^{\infty} 2^{2(k+1)} z^{2(k+1)}
\]

Taking inverse z-transform, we get
\[
x[n] = -\sum_{k=0}^{\infty} 2^{2(k+1)} \delta[n + 2(k + 1)]
\]
\[ z^{2(k+1)} \xrightarrow{Z^{-1}} \delta[n + 2(k + 1)] \]
**SOL 6.1.61** Option (A) is correct.

Using Taylor’s series expansion for a right-sided signal, we have

\[
\ln(1 + \alpha) = \alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{3} - \frac{\alpha^4}{4} + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\alpha)^k
\]

Thus,

\[
X(z) = \ln(1 + z^{-1}) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (z^{-1})^k
\]

Taking inverse \(z\)-transform we get

\[
x[n] = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \delta[n - k]
\]

**SOL 6.1.62** Option (D) is correct.

From the given pole-zero pattern

\[
X(z) = \frac{A z}{(z - \frac{1}{2})(z - 2)}, \quad A \rightarrow \text{Some constant}
\]

Using partial fraction expansion, we write

\[
X(z) = \frac{\alpha}{z - \frac{1}{2}} + \frac{\beta}{z - 2}, \quad \alpha \text{ and } \beta \text{ are constants.}
\]

Poles are at \(z = \frac{1}{2}\) and \(z = 2\). We obtain the inverse \(z\)-transform using relationship between the location of poles and region of convergence as shown in following figures.

**ROC : \(|z| > 2\)**

\[
\text{ROC is exterior to the circle passing through right most pole so both the terms in equation (1) corresponds to right sided sequences}
\]

\[
x[n] = \alpha \left(\frac{1}{2}\right)^n u[n] + \beta (2)^n u[n]
\]

**ROC : \(\frac{1}{3} < |z| < 2\)**
Since ROC has greater radius than the pole at \( z = \frac{1}{2} \), so first term in equation (1) corresponds to right-sided sequence

\[
\frac{\alpha}{1 - \frac{1}{2} z^{-1}} \rightarrow \alpha \left( \frac{1}{3} \right)^n u[n] \quad \text{(right-sided)}
\]

ROC \( |z| < 2 \) has radius less than the pole at \( z = 2 \), so the second term in equation (1) corresponds to left sided sequence.

\[
\frac{\beta}{1 - 2z^{-1}} \rightarrow \beta (2)^n u[n - 1] \quad \text{(left-sided)}
\]

So,

\[
x_2[n] = \alpha \left( \frac{1}{3} \right)^n u[n] + \beta (2)^n u[n - 1]
\]

**ROC :** \( |z| < \frac{1}{3} \)

ROC is left side to both the poles of \( X(z) \), so they corresponds to left sided signals.

\[
x_3[n] = \alpha \left( \frac{1}{3} \right)^n u[-n - 1] + \beta (2)^n u[-n - 1]
\]

All gives the same \( z \)-transform with different ROC, so all are the solution.

**SOL 6.1.63**

Option (C) is correct.

The \( z \)-transform of all the signal is same given as

\[
X(z) = \frac{1}{1 - 2z^{-1}} - \frac{1}{1 - \frac{1}{2} z^{-1}}
\]

Poles are at \( z = 2 \) and \( z = \frac{1}{2} \). Now consider the following relationship between ROC and location of poles.

1. Since \( x_1[n] \) is right-sided signal, so ROC is region in \( z \)-plane having radius greater than the magnitude of largest pole. So, \( |z| > 2 \) and \( |z| > \frac{1}{2} \) gives \( R_1 : |z| > 2 \)
2. Since \( x_2[n] \) is left-sided signal, so ROC is the region inside a circle having radius equal to magnitude of smallest pole. So, \( |z| < 2 \) and \( |z| < \frac{1}{2} \) gives \( R_2 : |z| < \frac{1}{2} \)
3. Since \( x_3[n] \) is double sided signal, so ROC is the region in \( z \)-plane such
as \( |z| > \frac{1}{2} \) and \( |z| < 2 \) which gives \( R_3 : \frac{1}{2} < |z| < 2 \)

**SOL 6.1.64**

Option (B) is correct.

We have

\[
X(z) = \frac{1 + \frac{1}{2}z^{-1}}{1 - \frac{1}{2}z^{-1}(1 + \frac{1}{3}z^{-1})} = \frac{2}{1 - \frac{1}{2}z^{-1}} + \frac{-1}{1 + \frac{1}{3}z^{-1}}
\]

\( X(z) \) has poles at \( z = \frac{1}{2} \) and \( z = -\frac{1}{3} \), we consider the different ROC’s and location of poles to obtain the inverse z-transform.

1. ROC \( |z| > \frac{1}{2} \) is exterior to the circle which passes through outtermost pole, so both the terms in equation (1) contributes to right sided sequences.

\[
x[n] = \frac{2}{2^n} u[n] - \left(-\frac{1}{3}\right)^n u[n]
\]

2. ROC \( |z| < \frac{1}{3} \) is interior to the circle passing through left most poles, so both the terms in equation (1) corresponds to left sided sequences.

\[
x[n] = \left[-\frac{2}{2^n} + \left(-\frac{1}{3}\right)^n\right] u[-n - 1]
\]

3. ROC \( \frac{1}{3} < |z| < \frac{1}{2} \) is interior to the circle passing through pole at \( z = \frac{1}{2} \) so the first term in equation (1) corresponds to a right sided sequence, while the ROC is exterior to the circle passing through pole at \( z = -\frac{1}{3} \), so the second term corresponds to a left sided sequence. Therefore, inverse z-transform is

\[
x[n] = \frac{-2}{2^n} u[-n - 1] - \left(-\frac{1}{3}\right)^n u[n]
\]

**SOL 6.1.65**

Option (A) is correct.

The location of poles and the ROC is shown in the figure. Since the ROC includes the point \( z = \frac{3}{4} \), ROC is \( \frac{1}{2} < |z| < 1 \)

\[
X(z) = \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 + z^{-1}}
\]

ROC is exterior to the pole at \( z = \frac{1}{2} \), so this term corresponds to a right-sided sequence, while ROC is interior to the pole at \( z = -1 \) so the second term corresponds to a left sided sequence. Taking inverse z-transform we get

\[
x[n] = \frac{A}{2^n} u[n] + B (-1)^n u[-n - 1]
\]

For \( n = 1 \),

\[
x[1] = \frac{A}{2} (1) + B \times 0 = 1 \Rightarrow \frac{A}{2} = 1 \Rightarrow A = 2
\]

For \( n = -1 \),

\[
x[-1] = A \times 0 + B (-1) = 1 \Rightarrow B = -1
\]

So,

\[
x[n] = \frac{1}{2^n} u[n] - (-1)^n u[-n - 1]
\]

**SOL 6.1.66**

Option (B) is correct.
Let, \[x[n] = C p^n u[n]\] (right-sided sequence having a single pole)
\[x[0] = 2 = C\]
\[x[2] = \frac{1}{2^2} = 2p^2 \Rightarrow p = \frac{1}{2},\]

So,
\[x[n] = 2 \left(\frac{1}{2}\right)^n u(n)\]

**SOL 6.1.67**

Option (D) is correct.

\[X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n + \sum_{n=\infty}^{1} \left(\frac{1}{4}z^{-1}\right)^n\]
\[= \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n + \sum_{m=1}^{\infty} (4z)^{-m} = \frac{1}{1 - \frac{1}{2}z^{-1}} - \frac{1}{1 - \frac{1}{4}z^{-1}}\]

**ROC:** Summation I converges if \(\left|\frac{1}{2}z^{-1}\right| < 1\) or \(|z| > \frac{1}{2}\) and summation II converges if \(4z^{-1} < 1\) or \(|z| < \frac{1}{4}\). 
ROC would be intersection of both which does not exist.

**SOL 6.1.68**

Option (C) is correct.

\[x[n] \rightarrow \frac{z^2}{z^2 - 16}, \quad \text{ROC } |z| < 4\]
\[x[n-2] \rightarrow z^{-2} \left(\frac{z^2}{z^2 - 16}\right) - \frac{1}{z^2 - 16} \quad \text{(Time shifting property)}\]

**SOL 6.1.69**

Option (B) is correct.

\[x[n] \rightarrow \frac{z^2}{z^2 - 16}, \quad \text{ROC } |z| < 4\]
\[\frac{1}{z^n} x[n] \rightarrow \frac{(2z)^2}{(2z)^2 - 16} = \frac{z^2}{z^2 - 4} \quad \text{(Scaling in } z\text{-domain)}\]

**SOL 6.1.70**

Option (C) is correct.

\[x[n] \rightarrow \frac{z^2}{z^2 - 16}, \quad \text{ROC } |z| < 4\]
\[x[-n] \rightarrow \frac{(\frac{1}{2})^2}{(\frac{1}{2})^2 - 16} \quad \text{(Time reversal property)}\]
\[x[-n] * x[n] \rightarrow \frac{(\frac{1}{2})^2 \left[\frac{z^2}{z^2 - 16}\right]}{257z^2 - 16z^4 - 16} \quad \text{(Time convolution property)}\]

**SOL 6.1.71**

Option (A) is correct.

\[x[n] \rightarrow \frac{z^2}{z^2 - 16}, \quad \text{ROC } |z| < 4\]
\[n x[n] \rightarrow -z \frac{d}{dz} \frac{z^2}{z^2 - 16} \quad \text{(Differentiation in } z\text{-domain)}\]
\[\rightarrow \frac{32z^2}{(z^2 - 16)^2}\]

**SOL 6.1.72**

Option (B) is correct.

\[x[n] \rightarrow \frac{z^2}{z^2 - 16}, \quad \text{ROC } |z| < 4\]
\[x[n+1] \rightarrow zX(z) \quad \text{(Time shifting)}\]
\[x[n-1] \rightarrow z^{-1}X(z) \quad \text{(Time shifting)}\]
\[x[n+1] + x[n-1] \rightarrow (z + z^{-1})X(z) \quad \text{(Linearity)}\]
\[ \frac{z}{z^2 - 16} \]

**SOL 6.1.73**

Option (D) is correct.

\[ x[n] \overset{z}{\rightarrow} \frac{z^2}{z^2 - 16}, \quad \text{ROC } |z| < 4 \]

\[ x[n - 3] \overset{z}{\rightarrow} z^3 \left(\frac{z^2}{z^2 - 16}\right) = \frac{z^2 - 1}{z^2 - 16} \quad (\text{Time shifting property}) \]

\[ x[n] \cdot x[n - 3] \overset{z}{\rightarrow} \left(\frac{z^2}{z^2 - 16}\right) \left(\frac{z^2 - 1}{z^2 - 16}\right) \quad (\text{Time convolution property}) \]

\[ \frac{z}{(z^2 - 16)^2} \]

**SOL 6.1.74**

Option (C) is correct.

We have,

\[ X(z) \overset{z^{-1}}{\rightarrow} 3^n n^2 u[n] \]

\[ X(2z) \overset{z^{-1}}{\rightarrow} \frac{1}{2z} \left(3^n n^2 u[n]\right) \quad (\text{Scaling in } z\text{-domain}) \]

**SOL 6.1.75**

Option (B) is correct.

\[ X(z) \overset{z^{-1}}{\rightarrow} 3^n n^2 u[n] \]

\[ X\left(\frac{1}{z}\right) \overset{z^{-1}}{\rightarrow} 3^{-\frac{n}{2}} (-n)^2 u[-n] \quad (\text{Time reversal}) \]

\[ \overset{z^{-1}}{\rightarrow} 3^n n^2 u[-n] \]

**SOL 6.1.76**

Option (C) is correct.

\[ X(z) \overset{z^{-1}}{\rightarrow} 3^n n^2 u[n] \]

\[ -z \frac{d}{dz} X(z) \overset{z^{-1}}{\rightarrow} n x[n] \quad (\text{Differentiation in } z\text{-domain}) \]

\[ z^{-1}\left[-z \frac{d}{dz} X(z)\right] \overset{z^{-1}}{\rightarrow} (n - 1) x[n - 1] \quad (\text{Time shifting}) \]

So,

\[ -z \frac{d}{dz} X(z) \overset{z^{-1}}{\rightarrow} -(n - 1) x[n - 1] \]

\[ -z^{-1}\left[-z \frac{d}{dz} X(z)\right] \overset{z^{-1}}{\rightarrow} -(n - 1)^3 3^{n - 1}(n - 1)^2 u[n - 1] \]

\[ \overset{z^{-1}}{\rightarrow} -(n - 1)^3 3^{n - 1} u[n - 1] \]

**SOL 6.1.77**

Option (A) is correct.

\[ \frac{1}{2} z^2 X(z) \overset{z^{-1}}{\rightarrow} \frac{1}{2} x[n + 2] \quad (\text{Time shifting}) \]

\[ \frac{1}{2} z^{-2} X(z) \overset{z^{-1}}{\rightarrow} \frac{1}{2} x[n - 2] \quad (\text{Time shifting}) \]

\[ \frac{z^2 - z^{-2}}{2} X(z) \overset{z^{-1}}{\rightarrow} \frac{1}{2} (x[n + 2] - x[n - 2]) \quad (\text{Linearity}) \]

**SOL 6.1.78**

Option (B) is correct.

\[ X(z) \overset{z^{-1}}{\rightarrow} 3^n n^2 u[n] \]

\[ X(z) X(z) \overset{z^{-1}}{\rightarrow} x[n] \cdot x[n] \quad (\text{Time convolution}) \]

**SOL 6.1.79**

Option (A) is correct.

\[ X(z) = 1 + \frac{z^{-1}}{4} \quad Y(z) = 1 - \frac{3z^{-1}}{4} \]

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{5}{2} + \frac{3}{2} z^{-1} - \frac{3}{2} z^{-2}}{1 + \frac{5}{2} z^{-1} + \frac{1}{4} z^{-2}} \]

For a causal system impulse response is obtained by taking right-sided inverse
z-transform of transfer function \( H(z) \). Therefore,

\[
h[n] = \frac{1}{3} \left[ 5 \left( -\frac{1}{2} \right)^n - 2 \left( \frac{1}{4} \right)^n \right] u[n]
\]

**SOL 6.1.80**

Option (D) is correct.

We have \( x[n] = (-3)^n u[n] \)

and

\( y[n] = \left[ 4(2)^n - \left( \frac{1}{2} \right)^n \right] u[n] \)

Taking z-transform of above we get

\[
X(z) = \frac{1}{1 + 3z^{-1}}
\]

and

\[
Y(z) = \frac{4}{1 - 2z^{-1}} - \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{3}{(1 - 2z^{-1})(1 - \frac{1}{2}z^{-1})}
\]

Thus transfer function is

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{10}{1 - 2z^{-1}} + \frac{7}{1 - \frac{1}{2}z^{-1}}
\]

For a causal system impulse response is obtained by taking right-sided inverse z-transform of transfer function \( H(z) \). Therefore,

\[
h[n] = \left[ 10(2)^n - 7\left( \frac{1}{2} \right)^n \right] u[n]
\]

**SOL 6.1.81**

Option (D) is correct.

We have \( h[n] = \left( \frac{1}{2} \right)^n u[n] \)

and

\( y[n] = 2\delta[n - 4] \)

Taking z-transform of above we get

\[
H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}
\]

and

\[
Y(z) = 2z^{-4}
\]

Now

\[
X(z) = \frac{Y(z)}{H(z)} = 2z^{-4} - z^{-5}
\]

Taking inverse z-transform we have

\[
x[n] = 2\delta[n - 4] - \delta[n - 5]
\]

**SOL 6.1.82**

Option (C) is correct.

We have,

\[
\]

Taking z-transform we get

\[
Y(z) = X(z) - z^{-2}X(z) + z^{-4}X(z) - z^{-6}X(z)
\]

or

\[
H(z) = \frac{Y(z)}{X(z)} = (1 - z^{-2} + z^{-4} - z^{-6})
\]

Taking inverse z-transform we have

\[
h[n] = \delta[n] - \delta[n - 2] + \delta[n - 4] - \delta[n - 6]
\]

**SOL 6.1.83**

Option (A) is correct.

We have

\[
h[n] = \frac{3}{4} \left( \frac{1}{4} \right)^{n-1} u[n - 1]
\]

Taking z-transform we get

\[
H(z) = \frac{Y(z)}{X(z)} = \left( \frac{\frac{3}{2}z^{-1}}{1 - \frac{1}{4}z^{-1}} \right) (\frac{1}{4})^{n-1} u[n - 1] \rightarrow z^{-1} \left( \frac{1}{1 - \frac{1}{4}z^{-1}} \right)
\]

or,

\[
Y(z) = \frac{1}{4} z^{-1} Y(z) = \frac{3}{4} z^{-1} X(z)
\]
Taking inverse z-transform we have
\[ y[n] - \frac{1}{4}y[n - 1] = \frac{3}{4}x[n - 1] \]

**SOL 6.1.84**

Option (A) is correct.

We have, \( h[n] = \delta[n] - \delta[n - 5] \)

Taking z-transform we get
\[ H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-5} \]

or
\[ Y(z) = X(z) - z^{-5}X(z) \]

Taking inverse z-transform we get
\[ y[n] = x[n] - x[n - 5] \]

**SOL 6.1.85**

Option (A) is correct.

Taking z transform of all system we get
\[
Y_1(z) = 0.2z^{-1}Y(z) + X(z) - 0.3z^{-1}X(z) + 0.02z^{-2}X(z) \\
H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1 - 0.3z^{-1} + 0.02z^{-2}}{1 - 0.2z^{-1}} \\
= \frac{(1 - 0.2z^{-1}) - (1 - 0.1z^{-1})}{(1 - 0.2z^{-1})} = (1 - 0.1z^{-1}) \\
Y_2(z) = X(z) - 0.1z^{-1}X(z) \\
H_2(z) = \frac{Y_2(z)}{X(z)} = (1 - 0.1z^{-1}) \\
Y_3(z) = 0.5z^{-1}Y(z) + 0.4X(z) - 0.3z^{-1}X(z) \\
H_3(z) = \frac{Y_3(z)}{X(z)} = \frac{0.4 - 0.3z^{-1}}{1 - 0.5z^{-1}} \\
H_1(z) = H_2(z), \text{ so } y_1 \text{ and } y_2 \text{ are equivalent.}
\]

**SOL 6.1.86**

Option (B) is correct.

We have
\[ H(z) = \frac{1}{1 - 2z^{-1}} + \frac{1}{1 + \frac{1}{2}z^{-1}} \]

Poles of \( H(z) \) are at \( z = 2 \) and \( z = -\frac{1}{2} \). Since \( h[n] \) is stable, so ROC includes unit circle \(|z| = 1\) and for the given function it must be \( \frac{1}{2} < |z| < 2 \). The location of poles and ROC is shown in the figure below.

**Consider the following two cases:**

1. ROC is interior to the circle passing through pole at \( z = 2 \), so this term corresponds to a left-sided signal.
\[
\frac{1}{1 - 2z^{-1}} \cdot z^{-n} \rightarrow (2)^nu[-n - 1] \quad \text{(left-sided)}
\]
2. ROC is exterior to the circle passing through pole at \( z = -\frac{1}{2} \), so this term corresponds to a right-sided signal.

\[
\frac{1}{1 + \frac{3}{2}z^{-1}} \cdot \left( -\frac{1}{z} \right)^n u[n]
\]

(right-sided)

Impulse response,

\[
h[n] = -(2)^n u[-n - 1] + \left( -\frac{1}{2} \right)^n u[n]
\]

SOL 6.1.87
Option (B) is correct.

We have

\[
H(z) = \frac{5z^2}{z^2 - z - 6} = \frac{5z^2}{(z - 3)(z + 2)}
\]

\[
= \frac{5}{(1 - 3z^{-1})(1 + 2z^{-1})} = \frac{3}{1 - 3z^{-1}} + \frac{2}{1 + 2z^{-1}}
\]

Since \( h[n] \) is causal, therefore impulse response is obtained by taking right-sided inverse \( z \)-transform of the transfer function \( X(z) \)

\[
h[n] = [3^{n+1} + 2(-2^n)]u[n]
\]

SOL 6.1.88
Option (D) is correct.

Zero at \( z = 0 \), \( z = \frac{2}{3} \), poles at \( z = \frac{1 \pm \sqrt{2}}{2} \)

(1) For a causal system all the poles of transfer function lies inside the unit circle \( |z| = 1 \). But, for the given system one of the pole does not lie inside the unit circle, so the system is not causal and stable.

(2) Not all poles and zero are inside unit circle \( |z| = 1 \), the system is not minimum phase.

SOL 6.1.89
Option (A) is correct.

\[
X(z) = \frac{-\frac{2}{3}}{1 - \frac{3}{4}z^{-1}} + \frac{\frac{27}{8}}{1 - 3z^{-1}}
\]

Poles are at \( z = \frac{1}{4} \) and \( z = 3 \). Since \( X(z) \) converges on \( |z| = 1 \), so ROC must include this circle. Thus for the given signal ROC : \( \frac{1}{4} < |z| < 3 \)

ROC is exterior to the circle passing through the pole at \( z = \frac{1}{4} \) so this term will have a right sided inverse \( z \)-transform. On the other hand ROC is interior to the circle passing through the pole at \( z = 3 \) so this term will have a left sided inverse \( z \)-transform.

\[
x[n] = -\frac{1}{3^n - 8} u[n] - \frac{3^{n+3}}{8} u[-n - 1]
\]

SOL 6.1.90
Option (C) is correct.

Since ROC includes the unit circle \( z = 1 \), therefore the system is both stable
and causal.

**SOL 6.1.91** Option (C) is correct.

1. Pole of system \( z = -\frac{1}{2}, \frac{1}{3} \) lies inside the unit circle \( |z| = 1 \), so the system is causal and stable.

2. Zero of system \( H(z) \) is \( z = -\frac{1}{2} \), therefore pole of the inverse system is at \( z = -\frac{1}{2} \) which lies inside the unit circle, therefore the inverse system is also causal and stable.

**SOL 6.1.92** Option (C) is correct.

Writing the equation from given block diagram we have

\[
[2Y(z) + X(z)]z^{-2} = Y(z)
\]

or

\[
H(z) = \frac{z^{-2}}{1 - 2z^{-1}} = -\frac{1}{2} + \frac{\frac{1}{2}}{1 - \sqrt{2}z^{-1}} + \frac{\frac{1}{4}}{1 + \sqrt{2}z^{-1}}
\]

Taking inverse laplace transform we have

\[
h[n] = -\frac{1}{2}\delta[n] + \frac{1}{4}\left((\sqrt{2})^n + (\sqrt{2})^n\right)u[n]
\]

**SOL 6.1.93** Option (D) is correct.

\[
Y(z) = X(z)z^{-1} - \{Y(z)z^{-1} + Y(z)z^2\}
\]

\[
X(z) = \frac{z^{-1}}{1 + z^{-1} + z^{-2}} = \frac{z}{z^2 + z + 1}
\]

So this is a solution but not unique. Many other correct diagrams can be drawn.

***********
SOLUTIONS 6.2

SOL 6.2.1 Correct answer is 2.

The z-transform is given as

\[ X(z) = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n z^{-n} = \sum_{n=0}^{\infty} \left( \frac{1}{2z} \right)^n \]

\[ = \frac{1}{1 - \frac{1}{2z}} = \frac{2z}{2z - 1} \]

...(1)

From the given question, we have

\[ X(z) = \frac{az}{az - 1} \]

...(2)

So, by comparing equations (1) and (2), we get \( a = 2 \).

SOL 6.2.2 Correct answer is \(-1.125\).

The z-transform of given sequence is

\[ X(z) = z^2 + z^2 - z^1 - z^0 \]

\[ = z^2 + z^2 - z - 1 \]

Now \[ X \left( \frac{1}{2} \right) = \left( \frac{1}{2} \right)^3 + \left( \frac{1}{2} \right)^2 - \left( \frac{1}{2} \right) - 1 = -1.125 \]

SOL 6.2.3 Correct answer is 3.

\[ X(z) = \frac{z+1}{z(z-1)} = \frac{1}{z} + \frac{2}{z-1} \]

By partial fraction

\[ \frac{1}{z} + 2z^{-1} \left( \frac{z}{z-1} \right) \]

Taking inverse z-transform

\[ x[n] = -\delta[n-1] + 2u[n-1] \]

\[ x[0] = -0 + 0 = 0 \]

\[ x[1] = -1 + 2 = 1 \]

\[ x[2] = -0 + 2 = 2 \]

Thus, we obtain

\[ x[0] + x[1] + x[2] = 3 \]

SOL 6.2.4 Correct answer is 3.

\[ x[n] = \alpha^n u[n] \]

Let,

\[ y[n] = x[n+3]u[n] = \alpha^{n+3}u[n+3]u[n] \]

\[ = \alpha^{n+3}u[n] \quad u[n+3]u[n] = u[n] \]

\[ Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} \alpha^{n+3}z^{-n}u[n] = \sum_{n=0}^{\infty} \alpha^{n+3}z^{-n} \]

\[ = \alpha^3 \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \alpha^3 \frac{1}{1-\alpha z^{-1}} = \alpha^3 \left( \frac{z}{z-\alpha} \right) \]

...(1)

From the given question, we have

\[ Y(z) = \alpha^k \left( \frac{z}{z-\alpha} \right) \]

...(2)

So, by comparing equations (1) and (2), we get \( k = 3 \).

NOTE:
Do not apply time shifting property directly because \( x[n] \) is a causal signal.
SOL 6.2.5  
Correct answer is 10.
We know that
\[ a^n u[n] \overset{\text{to}}{\rightarrow} \frac{z}{z-a} \]
\[ a^{n-10} u[n-10] \overset{\text{to}}{\rightarrow} \frac{z^{10} z}{z-a} \]  
(time shifting property)
So,
\[ x[n] = a^{n-10} u[n-10] \] ...(1)
From the given question, we have
\[ x[n] = a^{n-k} u[n-k] \] ...(2)
So, by comparing equations (1) and (2), we get
\[ k = 10 \]

SOL 6.2.6  
Correct answer is 9.
We know that
\[ a^n u[n] \overset{\text{to}}{\rightarrow} \frac{z}{z-a} \]
or
\[ 3^n u[n] \overset{\text{to}}{\rightarrow} \frac{z}{z-3} \]
\[ 3^{n-3} u[n-3] \overset{\text{to}}{\rightarrow} z^{3} \left( \frac{z}{z-3} \right) \]  
So
\[ x[n] = 3^{n-3} u[n-3] \]

SOL 6.2.7  
Correct answer is 2.
\[ y[n] = n[n+1] u[n] \]
\[ y[n] = n^2 u[n] + n u[n] \]
We know that
\[ u[n] \overset{\text{to}}{\rightarrow} \frac{z}{z-1} \]
Applying the property of differentiation in z-domain
If,
\[ x[n] \overset{\text{to}}{\rightarrow} X(z) \]
then,
\[ n x[n] \overset{\text{to}}{\rightarrow} - z \frac{d}{dz} X(z) \]
so,
\[ n u[n] \overset{\text{to}}{\rightarrow} - z \frac{d}{dz} \left( \frac{z}{z-1} \right) \]
or,
\[ n u[n] \overset{\text{to}}{\rightarrow} \frac{z}{(z-1)^2} \]
Again by applying the above property
\[ n (n u[n]) \overset{\text{to}}{\rightarrow} - z \frac{d}{dz} \left( \frac{z}{(z-1)^2} \right) \]
\[ n^2 u[n] \overset{\text{to}}{\rightarrow} \frac{z(z+1)}{(z-1)^3} \]
So
\[ Y(z) = \frac{z}{(z-1)^2} + \frac{z(z+1)}{(z-1)^3} = \frac{2z^2}{(z-1)^3} \] ...(1)
From the given question, we have
\[ x[n] = \frac{kz^k}{(z-1)^{k+1}} \] ...(2)
So, by comparing equations (1) and (2), we get
\[ k = 2 \]

SOL 6.2.8  
Correct answer is −1.
Given that
\[ X(z) = \log(1-2z), \mid z \mid < \frac{1}{2} \]
Differentiating
The Z-Transform

\[
\frac{dX(z)}{dz} = \frac{-2}{1 - 2z^{-1}} = z^{-1} - \frac{1}{2}z^{-2}
\]

or,

\[
\frac{zdX(z)}{dz} = \frac{1}{1 - \frac{1}{2}z^{-1}}
\]

From z-domain differentiation property

\[nx[n] \rightarrow -z\frac{dX(z)}{dz}\]

so,

\[nx[n] \rightarrow -\frac{1}{1 - \frac{1}{2}z^{-1}}\]

From standard z-transform pair, we have

\[\left(\frac{1}{z}\right)^n u[-n - 1] \rightarrow z^{-n}\]

Thus

\[nx[n] = \left(\frac{1}{z}\right)^n u[-n - 1]\]

or,

\[x[n] = \frac{1}{n}\left(\frac{1}{z}\right)^n u[-n - 1]\] ...(1)

From the given question, we have

\[x[n] = \frac{1}{n}\left(\frac{1}{2}\right)^n u[a - n]\] ...(2)

So, by comparing equations (1) and (2), we get \(a = -1\)

SOL 6.2.9

Correct answer is 2.

\[Y(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{k=-\infty}^{\infty} x[k]z^{-2k} = X(z^2)\]

Put \(\frac{n}{2} = k\) or \(n = 2k\) \(\cdots (1)\)

From the given question, we have

\[Y(z) = X(z^4)\] ...(2)

So, by comparing equations (1) and (2), we get \(k = 2\)

SOL 6.2.10

Correct answer is 0.

\[X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}\]

\[Y(z) = X(z^3) = \sum_{n=-\infty}^{\infty} x[n](z^3)^{-n} = \sum_{n=-\infty}^{\infty} x[n]z^{-3n} = \sum_{k=-\infty}^{\infty} x[k/3]z^{-k}\]

Put \(3n = k\) or \(n = k/3\)

Thus

\[y[n] = x[n/3]\]

\[y[n] = \{(-0.5)^{n/3}, n = 0, 3, 6, ..., 0, \text{ otherwise}\]

Thus \(y[4] = 0\)

SOL 6.2.11

Correct answer is –6.

By taking z-transform of \(x[n]\) and \(h[n]\)

\[H(z) = 1 + 2z^{-1} - z^{-3} + z^{-4}\]

\[X(z) = 1 + 3z^{-1} - z^{-2} - 2z^{-3}\]

From the convolution property of z-transform

\[Y(z) = H(z)X(z)\]

\[Y(z) = 1 + 5z^{-1} + 5z^{-2} - 5z^{-3} - 6z^{-4} + 4z^{-5} + z^{-6} - 2z^{-7}\]

Sequence is \(y[n]\) = \(\{1, 5, 5, -5, -6, 4, 1, -2\}\)
y[4] = -6

**SOL 6.2.12**

Correct answer is 0.5.

x[n] can be written as

\[ x[n] = \frac{1}{2} [u[n] + (-1)^n u[n]] \]

z-transform of x[n]

\[ X(z) = \frac{1}{2} \left[ \frac{1}{1 - z^{-1}} + \frac{1}{1 + z^{-1}} \right] \]

From final value theorem

\[ x(\infty) = \lim_{z \to 1} (z - 1) X(z) \]

\[ = \frac{1}{2} \lim_{z \to 1} (z - 1) \left[ \frac{z}{z - 1} + \frac{z}{z + 1} \right] \]

\[ = \frac{1}{2} \lim_{z \to 1} \left[ z + \frac{z(z - 1)}{(z + 1)} \right] \]

\[ = \frac{1}{2} (1) = 0.5 \]

**SOL 6.2.13**

Correct answer is 0.5.

From initial value theorem

\[ x[0] = \lim_{z \to \infty} X(z) \]

\[ = \lim_{z \to \infty} \frac{0.5z^2}{(z - 1)(z - 0.5)} \]

\[ = \lim_{z \to \infty} \frac{0.5}{(1 - \frac{1}{z})(1 - \frac{0.5}{z})} = 0.5 \]

**SOL 6.2.14**

Correct answer is -2.5.

Taking z transform of input and output

\[ X(z) = \frac{z}{z - 0.5} \]

\[ Y(z) = 1 - 2z^{-1} = \frac{z - 2}{z} \]

Transfer function of the filter

\[ H(z) = \frac{Y(z)}{X(z)} \]

\[ = \left( \frac{z - 2}{z} \right) \left( \frac{z - 0.5}{z} \right) = \frac{z^2 - 2.5z + 1}{z^2} \]

\[ = 1 - 2.5z^{-1} + z^{-2} \]

Taking inverse z-transform

\[ h[n] = \{1, -2.5, 1\} \]

Therefore

\[ h[1] = -2.5 \]

**SOL 6.2.15**

Correct answer is 4.

Comparing the given system realization with the generic first order direct form II realization
Difference equation for above realization is
\[ y[n] + a_1 y[n - 1] = b_0 x[n] + b_1 x[n - 1] \]
Here \( a_1 = -2, b_0 = 3, b_1 = 4 \)
So \( y[n] - 2y[n - 1] = 4x[n] + 3x[n - 1] \)
Taking z-transform on both sides
\[ Y(z) - 2z^{-1}Y(z) = 4X(z) + 3z^{-1}X(z) \]
Transfer function
\[ H(z) = \frac{Y(z)}{X(z)} = \frac{4 + 3z^{-1}}{1 - 2z^{-1}} = \frac{4z + 3}{z - 2} \] ...(1)
From the given question, we have
\[ H(z) = \frac{Y(z)}{X(z)} = \frac{k(z + 1) - 1}{z - 2} \] ...(2)
So, by comparing equations (1) and (2), we get
\[ k = 4 \]

**SOL 6.2.16**
Correct answer is 0.3333.
The z-transform of each system response
\[ H_1(z) = 1 + \frac{1}{2}z^{-1}, \quad H_2(z) = \frac{z}{z - \frac{1}{2}} \]
The overall system function
\[ H(z) = H_1(z)H_2(z) = \left(1 + \frac{1}{2}z^{-1}\right)\left(\frac{z}{z - \frac{1}{2}}\right) = \frac{z + \frac{1}{2}}{z - \frac{1}{2}} \]
Input,
\[ x[n] = \cos(n\pi) \]
So, \( z = -1 \) and \( H(z = -1) = \frac{-1 + \frac{1}{2}}{1 - \frac{1}{2}} = -\frac{1}{3} \)
Output of system
\[ y[n] = H(z = -1)x[n] = \frac{1}{3}\cos(n\pi) \] ...(1)
From the given question, we have
\[ y[n] = k\cos(n\pi) \] ...(2)
So, by comparing equations (1) and (2), we get
\[ k = \frac{1}{3} = 0.3333 \]

**SOL 6.2.17**
Correct answer is 1.
From the given block diagram
\[ Y(z) = \alpha z^{-1}X(z) + \alpha z^{-1}Y(z) \]
\[ Y(z)(1 - \alpha z^{-1}) = \alpha z^{-1}X(z) \]
Transfer function
\[ Y(z) \quad \frac{X(z)}{Z(z)} = \frac{\alpha z^{-1}}{1 - \alpha z^{-1}} \]
For stability poles at \( z = 1 \) must be inside the unit circle.
So
\[ |\alpha| < 1 \]

**SOL 6.2.18**
Correct answer is -2.
\[ X^+(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} \delta[n - 2]z^{-n} = z^{-2} \]
\[ \sum_{n=-\infty}^{\infty} f[n]\delta[n - n_0] = f[n_0] \]
From the given question, we have
\[ X^+(z) = z^{-2} \] ...(2)
So, by comparing equations (1) and (2), we get
\[ k = -2 \]

**SOL 6.2.19**
Correct answer is \(-1\).

\[ X^+(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} \] ...

(1)

From the given question, we have
\[ X^+(z) = \frac{1}{1+a/z} \] ...

(2)

So, by comparing equations (1) and (2), we get
\[ k = -1 \]

**SOL 6.2.20**
Correct answer is 0.0417.

We know that,
\[ \cos \alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \frac{\alpha^8}{8!} - \ldots \]
\[ = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \alpha^{2k} \]

Thus,
\[ X(z) = \cos(z^{-3}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(z^{-3})^{2k}, \quad |z| > 0 \]

Taking inverse \( z \)-transform we get
\[ x[n] = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \delta[n - 6k] \]

Now for \( n = 12 \) we get,
\[ 12 - 6k = 0 \Rightarrow k = 2 \]

Thus,
\[ x[12] = \frac{(-1)^2}{4!} = \frac{1}{24} = 0.0417 \]

**SOL 6.2.21**
Correct answer is 4.

For anticausal signal initial value theorem is given as,
\[ x[0] = \lim_{z \to \infty} x(z) = \lim_{z \to \frac{12 - 21z}{3 - 7z + 12z^2}} \frac{12}{3} = 4 \]

**SOL 6.2.22**
Correct answer is 1.12.

Taking \( z \)-transform on both sides
\[ Y(z) = cz^{-1}Y(z) - 0.12z^{-1}Y(z) + z^{-1}X(z) + z^{-2}X(z) \]

Transfer function,
\[ H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + z^{-2}}{1 - cz^{-1} + 0.12z^{-2}} = \frac{z + 1}{z^2 - cz + 0.12} \]

Poles of the system are
\[ z = \frac{c \pm \sqrt{c^2 - 0.48}}{2} \]

For stability poles should lie inside the unit circle, so \( |z| < 1 \)
\[ \left| \frac{c \pm \sqrt{c^2 - 0.48}}{2} \right| < 1 \]

Solving this inequality, we get \( |c| < 1.12 \).
### SOLUTIONS 6.3

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SOLUTIONS 6.4

SOL 6.4.1 Option (C) is correct. 
z-transform of \( x[n] \)
\[
X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} a^n u[n]z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}
\]

SOL 6.4.2 Option (B) is correct. 
We have \( x[n] = \sum_{k=0}^{\infty} \delta[n - k] \)
\[
X(z) = \sum_{k=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{\infty} \delta[n - k]z^{-n} \right]
\]
Since \( \delta[n - k] \) defined only for \( n = k \) so
\[
X(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - 1/z} = \frac{z}{z - 1}
\]

SOL 6.4.3 Option (A) is correct. 
We have \( f(nT) = a^{nT} \)
Taking z-transform we get
\[
F(z) = \sum_{n=0}^{\infty} a^{nT} z^{-n} = \sum_{n=0}^{\infty} (a^T)^n z^{-n} = \sum_{n=0}^{\infty} \left( \frac{a^T}{z} \right)^n = \frac{z}{z - a^T}
\]

SOL 6.4.4 Option ( ) is correct. 
SOL 6.4.5 Option (A) is correct. 
\( x[n] = b^n u[n] + b^{-n} u[-n - 1] \)
z-transform of \( x[n] \) is given as
\[
X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}
\]
\[
= \sum_{n=0}^{\infty} b^n u[n]z^{-n} + \sum_{n=0}^{\infty} b^{-n} u[-n - 1]z^{-n}
\]
\[
= \sum_{n=0}^{\infty} b^n z^{-n} + \sum_{n=-\infty}^{0} b^{-n} z^{-n}
\]
In second summation, let \( n = -m \)
\[
X(z) = \sum_{n=0}^{\infty} b^n z^{-n} + \sum_{m=1}^{\infty} b^m z^{-n}
\]
\[
= \sum_{n=0}^{\infty} (bz^{-1})^n + \sum_{m=1}^{\infty} (bz)^m
\]
Summation I converges, if \( |bz^{-1}| < 1 \) or \( |z| > |b| \)
Summation II converges, if \( |bz| < 1 \) or \( |z| < \frac{1}{|b|} \)
since \( |b| < 1 \) so from the above two conditions ROC : \( |z| < 1 \).

SOL 6.4.6 Option (B) is correct. 
z-transform of signal \( a^n u[n] \) is
The Z-Transform

\[ X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=-\infty}^{\infty} (az^{-1})^n \quad u[n] = 1, \, n \geq 0 \]

\[ = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \]

Similarly, z-transform of signal \( a^n u[-n-1] \) is

\[ X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} \]

\[ = -\sum_{n=-\infty}^{\infty} a^n z^{-n} \quad \therefore u[-n-1] = 1, \, n \leq -1 \]

Let \( n = -m \), then

\[ X(z) = -\sum_{m=1}^{\infty} a^{-m} z^{-m} = -\sum_{m=1}^{\infty} (a z^{-1})^{-m} \]

\[ = \frac{a^{-1} z^{-1}}{1 - a^{-1} z^{-1}} = \frac{z}{z - a} \]

z-transform of both the signal is same.

(A) is true

**ROC**: To obtain ROC we find the condition for convergences of \( X(z) \) for both the transform.

Summation I converges if \( |a^{-1} z| < 1 \) or \( |z| > |a| \), so ROC for \( a^n u[n] \) is \( |z| > |a| \)

Summation II converges if \( |a^{-1} z| < 1 \) or \( |z| < |a| \), so ROC for \( -a^n u[-n-1] \) is \( |z| < |a| \).

(R) is true, but (R) is NOT the correct explanation of (A).

**SOL 6.4.7**

Option (B) is correct.

z-transform of \( x[n] \) is defined as

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]

\[ = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-jn} \quad \text{Putting} \quad z = re^{j\phi} \]

z-transform exists if \( |X(z)| < \infty \)

\[ \sum_{n=-\infty}^{\infty} |x[n] r^{-n} e^{-jn}| < \infty \]

or

\[ \sum_{n=-\infty}^{\infty} |x[n] r^{-n}| < \infty \]

Thus, z-transform exists if \( x[n] r^{-n} \) is absolutely summable.

**SOL 6.4.8**

Option (A) is correct.

\[ x[n] = \left( \frac{1}{3} \right)^n u[n] - \left( \frac{1}{2} \right)^n u[-n-1] \]

Taking z transform we have

\[ X(z) = \sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^n z^{-n} - \sum_{n=-\infty}^{-1} \left( \frac{1}{2} \right)^n z^{-n} \]

\[ = \sum_{n=0}^{\infty} \left( \frac{1}{3} z^{-1} \right)^n - \sum_{n=-\infty}^{-1} \left( \frac{1}{2} z^{-1} \right)^n \]

First term gives

\[ \frac{1}{3} z^{-1} < 1 - \frac{1}{3} < |z| \]

Second term gives

\[ \frac{1}{2} z^{-1} > 1 - \frac{1}{2} > |z| \]

Thus its ROC is the common ROC of both terms. That is
\[
\frac{1}{3} < |z| < \frac{1}{2}
\]

SOL 6.4.9 Option (C) is correct.
Here
\[ x_1[n] = \left(\frac{5}{6}\right)^n u[n] \]
\[ X_1(z) = \frac{1}{1 - \frac{5}{6}z^{-1}} \quad \text{ROC} : R_1 \to |z| > \frac{5}{6} \]
\[ x_2[n] = -\left(\frac{6}{5}\right)^n u[-n - 1] \]
\[ X_1(z) = 1 - \frac{1}{1 - \frac{6}{5}z^{-1}} \quad \text{ROC} : R_2 \to |z| < \frac{6}{5} \]
Thus ROC of \( x_1[n] + x_2[n] \) is \( R_1 \cap R_2 \) which is \( \frac{5}{6} < |z| < \frac{6}{5} \)

SOL 6.4.10 Option (A) is correct.
\[ x[n] = 2^n u[n] \]
The \( z \)-transform of \( x[n] \)
\[ X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=\infty}^{\infty} 2^n u[n]z^{-n} \]
\[ = \sum_{n=0}^{\infty} (2z^{-1})^n = 1 + 2z^{-1} + (2z^{-1})^2 + ... \]
\[ = \frac{1}{1 - 2z^{-1}} \]
the above series converges if \( |2z^{-1}| < 1 \) or \( |z| > 2 \)

SOL 6.4.11 Option (A) is correct.
We have
\[ h[n] = u[n] \]
\[ H(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n \]
\( H(z) \) is convergent if
\[ \sum_{n=0}^{\infty} (z^{-1})^n < \infty \]
and this is possible when \( |z^{-1}| < 1 \). Thus ROC is \( |z^{-1}| < 1 \) or \( |z| > 1 \)

SOL 6.4.12 Option (B) is correct.
(Please refer to table 6.1 of the book "Gate Guide signals & Systems" by same authors)
\[ (A) \quad u[n] \xrightarrow{Z^{-1}} \frac{z}{z-1} \quad (A \to 3) \]
\[ (B) \quad \delta[n] \xrightarrow{Z^{-1}} 1 \quad (B \to 1) \]
\[ (C) \quad \sin \omega T \bigg|_{-nT}^{nT} \xrightarrow{Z^{-1}} \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1} \quad (C \to 4) \]
\[ (D) \quad \cos \omega T \bigg|_{-nT}^{nT} \xrightarrow{Z^{-1}} \frac{z - \cos \omega T}{z^2 - 2z \cos \omega T + 1} \quad (D \to 2) \]

SOL 6.4.13 Option (A) is correct.
Inverse \( z \)-transform of \( X(z) \) is given as
\[ x[n] = \frac{1}{2\pi T} \int X(z)z^{-n-1}dz \]

SOL 6.4.14 Option (B) is correct.
\[ H(z) = z^{-m} \]
so
\[ h[n] = \delta[n - m] \]
### SOL 6.4.15

Option (C) is correct.

We know that

\[ \alpha^n u[n] \propto z \frac{1}{1-\alpha z^{-1}} \]

For \( \alpha = 1 \),

\[ u[n] \propto z \frac{1}{1-z^{-1}} \]

### SOL 6.4.16

Option (B) is correct.

\[ X(z) = \frac{0.5}{1-2z^{-1}} \]

Since ROC includes unit circle, it is left handed system

\[ x[n] = -(0.5)(2)^{-n}u[-n-1] \]

\[ x(0) = 0 \]

If we apply initial value theorem

\[ x(0) = \lim_{z \to 1} X(z) = \lim_{z \to 1} \frac{0.5}{1-2z^{-1}} = 0.5 \]

That is wrong because here initial value theorem is not applicable because signal \( x[n] \) is defined for \( n < 0 \).

### SOL 6.4.17

Option (B) is correct.

\[
F(z) = \frac{1}{z+1} = 1 - \frac{z}{z+1} = 1 - \frac{1}{1+z^{-1}}
\]

so,

\[ f(k) = \delta(k) - (-1) \]

Thus

\[ (-1)^k \propto z \frac{1}{1+z} \]

### SOL 6.4.18

Option (C) is correct.

\[ X(z) = \frac{z(2z - \frac{5}{6})}{(z - \frac{1}{2})(z - \frac{1}{3})} \]

or

\[ X(z) = \frac{z}{(z - \frac{1}{2})} + \frac{\frac{z}{3}}{(z - \frac{1}{3})} \]

Poles of \( X(z) \) are at \( z = \frac{1}{2} \) and \( z = \frac{1}{3} \)

**ROC :** \( |z| > \frac{1}{2} \) Since ROC is outside to the outer most pole so both the terms in equation (1) corresponds to right sided sequence.

So,

\[ x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n] \quad (A \rightarrow 4) \]
The Z-Transform

ROC : $|z| < \frac{1}{3}$: Since ROC is inside to the innermost pole so both the terms in equation (1) corresponds to left sided signals.

So,

$$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \left(\frac{1}{3}\right)^n u[-n-1] \quad (D \rightarrow 2)$$

ROC : $\frac{1}{3} < |z| < \frac{2}{3}$: ROC is outside to the pole $z = \frac{1}{3}$, so the second term of equation (1) corresponds to a causal signal. ROC is inside to the pole at $z = \frac{1}{2}$, so First term of equation (1) corresponds to anticausal signal.

So,

$$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(\frac{1}{3}\right)^n u[n] \quad (C \rightarrow 1)$$

ROC : $|z| < \frac{1}{3}$ & $|z| > \frac{1}{2}$: ROC : $|z| < \frac{1}{3}$ is inside the pole at $z = \frac{1}{3}$ so second term of equation (1) corresponds to anticausal signal. On the other hand, ROC : $|z| > \frac{1}{2}$ is outside to the pole at $z = \frac{1}{2}$, so the first term in equation (1) corresponds to a causal signal.

So,

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{3}\right)^n u[-n-1] \quad (B \rightarrow 3)$$
Option (A) is correct.

Given,
\[ X(z) = \frac{z}{(z-2)(z-3)}, \quad |z| < 2 \]

\[ X(z) = \frac{1}{z-3} - \frac{1}{z-2} \]

By partial fraction

or,
\[ X(z) = \frac{z}{z-3} - \frac{z}{z-2} \]  \( \ldots (1) \)

Poles of \( X(z) \) are \( z = 2 \) and \( z = 3 \)

**ROC :** \( |z| < 2 \)

Since ROC is inside the innermost pole of \( X(z) \), both the terms in equation (1) correspond to anticausal signals.

\[ x[n] = -3^nu[-n-1] + 2^nu[-n-n] = (2^n - 3^n)u[-n-1] \]

Option (D) is correct.

Given that \( X(z) = \frac{z}{(z-a)^2}, \quad |z| > a \)

Residue of \( X(z) z^{-n} \) at \( z = a \) is

\[ \frac{d}{dz} (z-a)^2 X(z) z^{n-1} \bigg|_{z=a} = \frac{d}{dz} (z-a)^2 \frac{z}{(z-a)^2} z^{n-1} \bigg|_{z=a} = \frac{d}{dz} z^n \bigg|_{z=a} = nz^{n-1} \bigg|_{z=a} = na^{n-1} \]

Option (C) is correct.

\[ X(z) = \frac{\frac{1}{z}}{1-az^{-1}} + \frac{\frac{1}{z}}{1-bz^{-1}}, \quad \text{ROC : } |a| < |z| < |b| \]

Poles of the system are \( z = a \), \( z = b \)

**ROC :** \( |a| < |z| < |b| \)
Since ROC is outside to the pole at \( z = a \), therefore the first term in \( X(z) \) corresponds to a causal signal.

\[
\frac{1}{1 - az^{-1}} = z^{-1} + \frac{1}{2}(a)^n u[n]
\]

ROC is inside to the pole at \( z = b \), so the second term in \( X(z) \) corresponds to an anticausal signal.

\[
\frac{1}{1 - bz^{-1}} = \frac{z^{-1}}{3} - \frac{1}{3}(b)^n u[-n - 1]
\]

\[
x[n] = \frac{1}{2}(a)^n u[n] - \frac{1}{3}(b)^n u[-n - 1]
\]

\[
x[0] = \frac{1}{2} u[0] - \frac{1}{3} u[-1] = \frac{1}{2}
\]

**SOL 6.4.22** Option (A) is correct.

\[
X(z) = \frac{(z + z^{-3})}{(z + z^{-3})} = \frac{z(1 + z^{-4})}{z(1 + z^{-2})}
\]

\[
= (1 + z^{-4})(1 + z^{-2})^{-1}
\]

Writing binomial expansion of \( (1 + z^{-2})^{-1} \), we have

\[
X(z) = (1 + z^{-4})(1 - z^{-2} + z^{-4} - z^{-6} + ...)
\]

\[
= 1 - z^{-2} + 2z^{-4} - 2z^{-6} + ...
\]

For a sequence \( x[n] \), its \( z \)-transform is

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}
\]

Comparing above two

\[
x[n] = \delta[n] - \delta[n - 2] + 2\delta[n - 4] - 2\delta[n - 6] + ...
\]

\[
x[n] = \{1, 0, -1, 0, 2, 0, -2, ..., \}
\]

\( x[n] \) has alternate zeros.

**SOL 6.4.23** Option (A) is correct.

We know that \( \alpha z^{\pm a} \xrightarrow{Z^{-1}} \alpha \delta[n \pm a] \)

Given that

\[
X(z) = 5z^2 + 4z^{-1} + 3
\]

Inverse \( z \)-transform

\[
x[n] = 5\delta[n + 2] + 4\delta[n - 1] + 3\delta[n]
\]

**SOL 6.4.24** Option (A) is correct.

\[
X(z) = e^{\frac{1}{2}z}
\]

\[
x[n] = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + \frac{x[1]}{2} + \frac{x[2]}{3} + \frac{x[3]}{4} + ...
\]

Comparing above two

\[
\{x[0], x[1], x[2], x[3]..., \} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \}
\]

\[
x[n] = \frac{1}{n} u[n]
\]

**SOL 6.4.25** Option (D) is correct.

The ROC of addition or subtraction of two functions \( x_1[n] \) and \( x_2[n] \) is \( R_1 \cap R_2 \). We have been given ROC of addition of two function and has been asked ROC of subtraction of two function. It will be same.
SOL 6.4.26
Option (D) is correct.

(A) 

\[ x[n] = \alpha^n u[n] \]

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} \quad u[n] = 1, \quad n \geq 0 \]

\[ = \sum_{n=0}^{\infty} (\alpha z^{-1})^n \]

\[ = \frac{1}{1 - \alpha z^{-1}}, \quad |\alpha z^{-1}| < 1 \quad \text{or} \quad |z| > |\alpha| \]  \quad \text{(A \to 2)}

(B) 

\[ x[n] = -\alpha^n u[-n - 1] \]

\[ X(z) = -\sum_{n=-\infty}^{\infty} \alpha^n u[-n - 1]z^{-n} \]

\[ = -\sum_{n=1}^{\infty} \alpha^n z^{-n} \quad u[-n - 1] = 1, \quad n \leq -1 \]

Let \( n = -m \),

\[ X(z) = -\sum_{m=1}^{\infty} \alpha^{-m} z^{-m} = -\sum_{m=1}^{\infty} (\alpha^{-1} z)^{-m} \]

\[ = -\frac{\alpha^{-1} z}{1 - \alpha^{-1} z}, \quad |\alpha^{-1} z| < 1 \quad \text{or} \quad |z| < |\alpha| \]

\[ = \frac{1}{(1 - \alpha z^{-1})}, \quad |z| < |\alpha| \]  \quad \text{(B \to 3)}

(C) 

\[ x[n] = -n\alpha^n u[-n - 1] \]

We have,

\[ -\alpha^n u[-n - 1] \xrightarrow{z^{-1}} \frac{1}{(1 - \alpha z^{-1})}, \quad |z| < |\alpha| \]

From the property of differentiation in \( z \)-domain

\[ -n\alpha^n u[-n - 1] \xrightarrow{z^{-1}} -z \frac{d}{dz} \left[ \frac{1}{1 - \alpha z^{-1}} \right], \quad |z| < |\alpha| \]

\[ = \frac{\alpha^{-1} z}{(1 - \alpha^{-1} z)^2}, \quad |z| < |\alpha| \]  \quad \text{(C \to 4)}

(D) 

\[ x[n] = n\alpha^n u[n] \]

We have,

\[ \alpha^n u[n] \xrightarrow{z^{-1}} \frac{1}{(1 - \alpha z^{-1})}, \quad |z| > |\alpha| \]

From the property of differentiation in \( z \)-domain

\[ n\alpha^n u[n] \xrightarrow{z^{-1}} -z \frac{d}{dz} \left[ \frac{1}{1 - \alpha z^{-1}} \right], \quad |z| > |\alpha| \]

\[ = \frac{\alpha^{-1} z}{(1 - \alpha^{-1} z)^2}, \quad |z| > |\alpha| \]  \quad \text{(D \to 1)}

SOL 6.4.27
Given that, \( z \) transform of \( x[n] \) is

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

\( z \)-transform of \( \{ x[ne^{-jn}] \} \)

\[ Y(z) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn}z^{-n} = \sum_{n=-\infty}^{\infty} x[n](ze^{-j\omega})^{-n} = X(z)|_{z=ze^{j\omega}} \]

so,

\[ Y(z) = X(ze^{-j\omega}) \]

SOL 6.4.28
Option (A) is correct.

SOL 6.4.29
Option (C) is correct.

We know that,

\[ a^n u[n] \xrightarrow{z^{-1}} \frac{z}{z-a} \]  \quad \text{(A \to 3)}

From time shifting property
From the property of scaling in $z$-domain
If, $x[n] \rightarrow X(z)$
then, $a^n x[n] \rightarrow X(z/a)$

so $(e^a)^n a^n \rightarrow \left(\frac{z}{e^a}\right)^n = \frac{ze^{-a}}{z^{e^{-a}}}$

From the property of differentiation in $z$-domain
If, $a^n u[n] \rightarrow \frac{z}{z-a}$
then, $n a^n u[n] \rightarrow \frac{dz}{dz} \left(\frac{z}{z-a}\right) = \frac{az}{(z-a)^2}$

SOL 6.4.30
Option (C) is correct.

The convolution of a signal $x[n]$ with unit step function $u[n]$ is given by

$$y[n] = x[n] \ast u[n] = \sum_{k=-\infty}^{\infty} x[k]$$

Taking $z$-transform

$$Y(z) = X(z) \frac{1}{1-z^{-1}}$$

SOL 6.4.31
Option (B) is correct.

From the property of $z$-transform,

$$x_1[n] \ast x_2[n] \rightarrow \frac{z}{z-1} X_1(z) X_2(z)$$

SOL 6.4.32
Option (C) is correct.

Given $z$ transform

$$C(z) = \frac{z^{-1}(1 - z^{-4})}{4(1 - z^{-1})^2}$$

Applying final value theorem

$$\lim_{n \to \infty} f(n) = \lim_{z \to 1} (z-1)f(z)$$

$$\lim_{z \to 1} (z-1)f(z) = \lim_{z \to 1} \frac{z^{-1}(1 - z^{-4})}{4(1 - z^{-1})^2}$$

$$= \lim_{z \to 1} \frac{z^{-1}(1 - z^{-4})(z-1)}{4(z-1)^2}$$

$$= \lim_{z \to 1} \frac{z^{-1}z-^4(z-1)(z-1)}{4z-^2(z-1)^2}$$

$$= \lim_{z \to 1} \frac{z^3(z-1)(z+1)(z^2+1)(z-1)}{4(z-1)^2}$$

$$= \lim_{z \to 1} \frac{z^3(z+1)(z^2+1)}{4} = 1$$

SOL 6.4.33
Option (C) is correct.

$$H_1(z) = 1 + 1.5z^{-1} - z^{-2}$$

$$= 1 + \frac{3}{2z} - \frac{1}{z^2} = \frac{2z^2 + 3z - 2}{2z^2}$$

Poles $z^2 = 0 \Rightarrow z = 0$

zeros $(2z^2 + 3z - 2) = 0 \Rightarrow (z - \frac{1}{2})(z + 2) = 0 \Rightarrow z = \frac{1}{2}, z = -2$

zeros of the two systems are identical.
Option (D) is correct.
Taking $z$-transform on both sides of given equation.
\[ z^2Y(z) + 6z^2Y(z) + 11zY(z) + 6Y(z) = z^2R(z) + 9zR(z) + 20R(z) \]
Transfer function
\[ \frac{Y(z)}{R(z)} = \frac{z^2 + 9z + 20}{z^2 + 6z + 11z + 6} \]

Option (A) is correct.
Characteristic equation of the system
\[ |zI - A| = 0 \]
\[ |zI - A| = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & -b \end{bmatrix} = \begin{bmatrix} z & -1 \\ -b & z + b \end{bmatrix} \]
\[ |zI - A| = z(z + b) + b = 0 \]
\[ z^2 + za + b = 0 \]
In the given options, only option (A) satisfies this characteristic equation.
\[ c[k + 2] + \alpha c[k + 1] + \beta c[k] = u[k] \]
\[ z^2 + za + b = 0 \]

Option (B) is correct.
We can see that the given impulse response is decaying exponential, i.e.
\[ h[n] = a^n u[n], \quad 0 < a < 1 \]
$z$-transform of $h[n]$
\[ H(z) = \frac{z}{z-a} \]
Pole of the transfer function is at $z = a$, which is on real axis between $0 < a < 1$.

Option (A) is correct.
Taking $z$-transform
\[ Y(z) + z^{-1}Y(z) = X(z) - z^{-1}X(z) \]
\[ Y(z) = \frac{(1 - z^{-1})}{(1 + z^{-1})} \]
which has a linear phase response.

Option (A) is correct.
For the linear phase response output is the delayed version of input multiplied by a constant.
\[ y[n] = kx[n - n_0] \]
\[ Y(z) = \frac{z^{-n_0}X(z)}{z^n} \]
Pole lies at $z = 0$

Option (B) is correct.
Given impulse response can be expressed in mathematical form as
\[ h[n] = \delta[n] - \delta[n - 1] + \delta[n - 2] - \delta[n - 3] + \ldots \]
By taking $z$-transform
\[ H(z) = 1 - z^{-1} + z^{-2} - z^{-3} + z^{-4} - z^{-5} + \ldots = (1 + z^{-2} + z^{-4} + \ldots) - (z^{-1} + z^{-3} + z^{-5} + \ldots) \]
\[ = \frac{1}{1-z^{-2}} - \frac{z^{-1}}{1-z^{-2}} = \frac{z^2}{z^2 - 1} - \frac{z}{z^2 - 1} \]
\[ = \frac{(z^2 - z)}{(z^2 - 1)(z - 1)(z + 1)} = \frac{z}{z + 1} \quad \text{Pole at } z = -1 \]
**SOL 6.4.41** Option (B) is correct.

Let Impulse response of system $h[n] \xrightarrow{z} H(z)$.

First consider the case when input is unit step.

**Input,**

$$x_1[n] = u[n] \text{ or } X_1(z) = \frac{z}{(z-1)}$$

**Output,**

$$y_1[n] = \delta[n] \text{ or } Y_1(z) = 1$$

so,

$$Y_1(z) = X_1(z) H(z)$$

$$1 = \frac{z}{(z-1)} H(z)$$

**Transfer function,**

$$H(z) = \frac{(z-1)}{z}$$

Now input is ramp function

$$x_2[n] = nu[n]$$

$$X_2(z) = \frac{z}{(z-1)^2}$$

**Output,**

$$Y_2(z) = X_2(z) H(z)$$

$$= \left[ \frac{z}{(z-1)^2} \right] \left[ \frac{(z-1)}{z} \right] = \frac{1}{(z-1)}$$

$$Y_2(z) \xrightarrow{z^{-1}} y_2[n] = \frac{1}{(z-1)} \xrightarrow{z^{-1}} u[n-1]$$

**SOL 6.4.42** Option (C) is correct.

Given state equations

$$s[n+1] = As[n] + B x[n] \quad \ldots(1)$$

$$y[n] = Cs[n] + D x[n] \quad \ldots(2)$$

Taking $z$-transform of equation (1)

$$zS(z) = A S(z) + B X(z)$$

$$S(z) [zI - A] = B X(z)$$

$$S(z) = (zI - A)^{-1} B X(z) \quad \ldots(3)$$

Now, taking $z$-transform of equation (2)

$$Y(z) = CS(z) + DX(z)$$

Substituting $S(z)$ from equation (3), we get

$$Y(z) = C (zI - A)^{-1} B X(z) + DX(z)$$

Transfer function

$$H(z) = \frac{Y(z)}{X(z)} = C (zI - A)^{-1} B + D$$

**SOL 6.4.43** Option (B) is correct.

$$F(z) = 4z^2 - 8z^2 - z + 2$$

$$F(z) = 4z^2(z - 2) - z(z - 2)$$

$$= (4z^2 - 2)(z - 2)$$

$$4z^2 - 2 = 0 \text{ and } (z - 2) = 0$$

$$z = \pm \frac{1}{2} \text{ and } z = 2$$

Only one root lies outside the unit circle.

**SOL 6.4.44** Option (A) is correct.

We know that convolution of $x[n]$ with unit step function $u[n]$ is given by

$$x[n] * u[n] = \sum_{k=-\infty}^{n} x[k]$$
So, \( y[n] = x[n]* u[n] \)

Taking z-transform on both sides,
\[
Y(z) = X(z) \frac{z}{(z-1)} = X(z) \frac{1}{(1-z^{-1})}
\]

Transfer function,
\[
H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1-z^{-1})}
\]

Now consider the inverse system of \( H(z) \), let impulse response of the inverse system is given by \( H_1(z) \), then we can write
\[
H(z)H_1(z) = 1
\]
\[
H_1(z) = \frac{X(z)}{Y(z)} = 1 - z^{-1}
\]
\[
(1-z^{-1})Y(z) = X(z)
\]
\[
Y(z) - z^{-1}Y(z) = X(z)
\]

Taking inverse z-transform
\[
y[n] - y[n-1] = x[n]
\]

SOL 6.4.45 Option (B) is correct.
\[
y[n] - \frac{1}{2}y[n-1] = x[n]
\]

Taking z-transform on both sides
\[
Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z)
\]

Transfer function
\[
H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}}
\]

Now, for input \( x[n] = k\delta[n] \) output is
\[
Y(z) = H(z)X(z)
\]
\[
= \frac{k}{1 - \frac{1}{2}z^{-1}}
\]

Taking inverse z-transform
\[
y[n] = k \left( \frac{1}{2} \right)^n u[n] = k \left( \frac{1}{2} \right)^n, \quad n \geq 0
\]

SOL 6.4.46 Option (A) is correct.
\[
y[n] + y[n-1] = x[n]
\]

For unit step response, \( x[n] = u[n] \)
\[
y[n] + y[n-1] = u[n]
\]

Taking z-transform
\[
Y(z) + z^{-1}Y(z) = \frac{z}{z-1}
\]
\[
(1+z^{-1})Y(z) = \frac{z}{z-1}
\]
\[
\frac{(1+z)}{z}Y(z) = \frac{z}{z-1}
\]
\[
Y(z) = \frac{z^2}{(z+1)(z-1)}
\]

SOL 6.4.47 Option (A) is correct.

SOL 6.4.48 Option (A) is correct.

SOL 6.4.49 Option (A) is correct.

We have \( h(2) = 1, h(3) = -1 \) otherwise \( h[k] = 0 \). The diagram of response
Sample Chapter of *Signals and Systems* (Vol-7, GATE Study Package)

is as follows:

```
0  2  3  k
-1
```

It has the finite magnitude values. So it is a finite impulse response filter. Thus $S_2$ is true but it is not a low pass filter. So $S_1$ is false.

**SOL 6.4.50**

Option (D) is correct.

$$H(z) = \frac{z}{z - 0.2}$$

We know that

$$-a^n u[-n - 1] \rightarrow \frac{1}{1 - az^{-1}}$$

Thus $$h[n] = -(0.2)^n u[-n - 1]$$

**SOL 6.4.51**

Option (B) is correct.

We have $$h[n] = 3\delta[n - 3]$$
or

$$H(z) = 2z^{-3}$$

Taking $z$ transform

$$X(z) = z^4 + z^2 - 2z - 3$$

Now

$$Y(z) = H(z)X(z)$$

$$= 2z^{-3}(z^4 + z^2 - 2z - 3)$$

$$= 2(z^4 + z^2 - 2z - 3)z^{-3}$$

Taking inverse $z$ transform we have

$$y[n] = 2\delta[n + 1] + \delta[n - 1] - 2\delta[n - 2] + 2\delta[n - 3] - 3\delta[n - 7]$$

At $n = 4$,

$$y[4] = 0$$

**SOL 6.4.52**

Option (A) is correct.

$z$-transform of $x[n]$ is

$$X(z) = 4z^{-3} + 3z^{-1} + 2 - 6z^2 + 2z^3$$

Transfer function of the system

$$H(z) = 3z^{-1} - 2$$

Output, $Y(z) = H(z)X(z)$

$$= (3z^{-1} - 2)(4z^{-3} + 3z^{-1} + 2 - 6z^2 + 2z^3)$$

$$= 12z^{-4} + 9z^{-2} + 6z^{-1} - 18z + 6z^2 - 8z^3 - 6z^4 - 4 + 12z^2 - 4z^3$$

$$= 12z^{-4} - 8z^{-2} + 4 - 18z + 18z^2 - 4z^3$$

Or sequence $y[n]$ is

$$y[n] = 12\delta[n - 4] - 8\delta[n - 3] + 9\delta[n - 2] - 4\delta[n] - 18\delta[n + 1] + 18\delta[n + 2] - 4\delta[n + 3]$$

$$y[n] \neq 0, n < 0$$

So $y[n]$ is non-causal with finite support.

**SOL 6.4.53**

Option (C) is correct.

Impulse response of given LTI system.

$$h[n] = x[n - 1] \ast y[n]$$
Taking z-transform on both sides.

$$H(z) = z^{-1}X(z)Y(z)$$

We have $X(z) = 1 - 3z^{-1}$ and $Y(z) = 1 + 2z^{-2}$

So

$$H(z) = z^{-1}(1 - 3z^{-1})(1 + 2z^{-2})$$

Output of the system for input $u[n] = \delta[n - 1]$ is,

$$y(z) = H(z)U(z)$$

So

$$Y(z) = z^{-1}(1 - 3z^{-1})(1 + 2z^{-2})z^{-1}$$

$$= z^{-2}(1 - 3z^{-1} + 2z^{-2} - 6z^{-3})$$

$$= z^{-2} - 3z^{-3} + 2z^{-4} - 6z^{-5}$$

Taking inverse z-transform on both sides we have output.

$$y[n] = \delta[n - 2] - 3\delta[n - 3] + 2\delta[n - 4] - 6\delta[n - 5]$$

**SOL 6.4.54**

Option (D) is correct.

$$H(z) = (1 - az^{-1})$$

We have to obtain inverse system of $H(z)$. Let inverse system has response $H_1(z)$.

$$H(z)H_1(z) = 1$$

$$H_1(z) = \frac{1}{H(z)} = \frac{1}{1 - az^{-1}}$$

For stability $H(z) = (1 - az^{-1})$, $|z| > a$, but in the inverse system $|z| < a$, for stability of $H_1(z)$.

so

$$h_1[n] = -a^n u[-n - 1]$$

**SOL 6.4.55**

Option (C) is correct.

$$H(z) = \frac{z}{z + \frac{1}{2}}$$

Pole,

$$z = -\frac{1}{2}$$

The system is stable if pole lies inside the unit circle. Thus (A) is true, (R) is false.

**SOL 6.4.56**

Option (A) is correct.

Difference equation of the system.

$$y[n + 2] - 5y[n + 1] + 6y[n] = x[n]$$

Taking z-transform on both sides of above equation.

$$z^2Y(z) - 5zY(z) + 6Y(z) = X(z)$$

$$(z^2 - 5z + 6)y(z) = X(z)$$

Transfer function,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(z^2 - 5z + 6)} = \frac{1}{(z - 2)(z - 3)}$$

Roots of the characteristic equation are $z = 2$ and $z = 3$

We know that an LTI system is unstable if poles of its transfer function (roots of characteristic equation) lies outside the unit circle. Since, for the given system the roots of characteristic equation lies outside the unit circle ($z = 2, z = 3$) so the system is unstable.

**SOL 6.4.57**

Option (C) is correct.

System function,

$$H(z) = \frac{z^2 + 1}{(z + 0.5)(z - 0.5)}$$

Poles of the system lies at $z = 0.5, z = -0.5$. Since, poles are within the unit
circle, therefore the system is stable. From the initial value theorem

\[ h[0] = \lim_{z \to \infty} H(z) = \lim_{z \to \infty} \frac{(z^2 + 1)}{(z + 0.5)(z - 0.5)} \]

\[ = \lim_{z \to \infty} \frac{\left(1 + \frac{1}{z}\right)}{\left(1 + \frac{0.5}{z}\right)\left(1 - \frac{0.5}{z}\right)} = 1 \]

SOL 6.4.58 Option (D) is correct.

\[ y[n] = 2x[n] + 4x[n - 1] \]

Taking z-transform on both sides

\[ Y(z) = 2X(z) + 4z^{-1}X(z) \]

Transfer Function,

\[ H(z) = \frac{Y(z)}{X(z)} = 2 + 4z^{-1} = \frac{2z + 4}{z} \]

Pole of \( H(z) \), \( z = 0 \)

Since Pole of \( H(z) \) lies inside the unit circle so the system is stable. (A) is not True.

Taking inverse z-transform

\[ h[n] = 2\delta[n] + 4\delta[n - 1] = \{2, 4\} \]

Impulse response has finite number of non-zero samples. (R) is true.

SOL 6.4.59 Option (B) is correct.

For left sided sequence we have

\[ -a^n u[-n - 1] \overset{z}{\longrightarrow} \frac{1}{1 - az^{-1}} \quad \text{where} \quad |z| < a \]

Thus

\[ -5^n u[-n - 1] \overset{z}{\longrightarrow} \frac{1}{1 - 5z^{-1}} \quad \text{where} \quad |z| < 5 \]  

or

\[ -5^n u[-n - 1] \overset{z}{\longrightarrow} \frac{z}{z - 5} \quad \text{where} \quad |z| < 5 \]

Since ROC is \(|z| < 5\) and it include unit circle, system is stable.

ALTERNATIVE METHOD :

\[ h[n] = -5^n u[-n - 1] \]

\[ H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=-\infty}^{\infty} -5^n z^{-n} = -\sum_{n=-\infty}^{\infty} (5z^{-1})^n \]

Let \( n = -m \), then

\[ H(z) = -\sum_{m=1}^{\infty} (5z^{-1})^{-m} = 1 - \sum_{m=0}^{\infty} (5z^{-1})^{-m} \]

\[ = 1 - \frac{1}{\frac{1}{5}z^{-1}} = 1 - \frac{5}{z - 5} = \frac{z}{z - 5} \]

SOL 6.4.60 Option (B) is correct.

For a system to be stable poles of its transfer function \( H(z) \) must lie inside the unit circle. In inverse system poles will appear as zeros, so zeros must be inside the unit circle.

SOL 6.4.61 Option (C) is correct.

A n LTI discrete system is said to be BIBO stable if its impulse response \( h[n] \)
is summable, that is
\[ \sum_{n=-\infty}^{\infty} h[n] < \infty \]

z-transform of \( h[n] \) is given as
\[ H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \]

Let \( z = e^{j\omega} \) (which describes a unit circle in the z-plane), then
\[ |H(z)| = \left| \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \right| \\
= \sum_{n=-\infty}^{\infty} |h[n]|e^{-j\omega n} \\
= \sum_{n=-\infty}^{\infty} |h[n]| < \infty 
\]

which is the condition of stability. So LTI system is stable if ROC of its system function includes the unit circle \( |z| = 1 \).

(A) is true.

We know that for a causal system, the ROC is outside the outermost pole. For the system to be stable ROC should include the unit circle \( |z| = 1 \). Thus, for a system to be causal & stable these two conditions are satisfied if all the poles are within the unit circle in z-plane.

(R) is false.

SOL 6.4.62
Option (B) is correct.

We know that for a causal system, the ROC is outside the outermost pole. For the system to be stable ROC should include the unit circle \( |z| = 1 \). Thus, for a system to be causal & stable these two conditions are satisfied if all the poles are within the unit circle in z-plane.

(A) is true.

If the z-transform \( X(z) \) of \( x[n] \) is rational then its ROC is bounded by poles because at poles \( X(z) \) tends to infinity.

(R) is true but (R) is not correct explanation of (A).

SOL 6.4.63
Option (C) is correct.

We have,
\[ H(z) = \frac{2 - \frac{3}{2}z^{-1}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{1}{1 - \frac{3}{2}z^{-1}} + \frac{1}{(1 - \frac{1}{2}z^{-1})} \]

By partial fraction

For ROC : \( |z| > 1/2 \)
\[ h[n] = \left( \frac{1}{2} \right)^n u[n] + \left( \frac{1}{4} \right)^n u[n], \ n > 0 \]
\[ |1/z| = a^n u[n], |z| > a \]

Thus, system is causal. Since ROC of \( H(z) \) includes unit circle, so it is stable also. Hence \( S_1 \) is True.

For ROC : \( |z| < 1/4 \)
\[ h[n] = -\left( \frac{1}{2} \right)^n u[-n-1] + \left( \frac{1}{4} \right)^n u[n], \ |z| > 1/4 \]

System is not causal. ROC of \( H(z) \) does not include unity circle, so it is not stable and \( S_3 \) is True.

SOL 6.4.64
Option (C) is correct.

We have \( 2y[n] = a y[n-2] - 2x[n] + bx[n-1] \)
Taking z transform we get

\[ 2Y(z) = \alpha Y(z) z^{-2} - 2X(z) + \beta X(z) z^{-1} \]

or

\[ Y(z) = \left( \frac{\beta z^{-1} - 2}{2 - \alpha z^{-2}} \right) X(z) \]  

or

\[ H(z) = \frac{z(\beta z^{-1} - 2)}{(z^2 - \alpha z^{-2})} \]

It has poles at \( \pm \sqrt{\alpha/2} \) and zero at 0 and \( \beta /2 \). For a stable system poles must lie inside the unit circle of z plane. Thus

\[ |\sqrt{\frac{\alpha}{2}}| < 1 \]

or

\[ |\alpha| < 2 \]

But zero can lie anywhere in plane. Thus, \( \beta \) can be of any value.

**SOL 6.4.65**

Option (D) is correct.

Let \( H_1(z) \) and \( H_2(z) \) are the transfer functions of systems \( s_1 \) and \( s_2 \) respectively.

For the second order system, transfer function has the following form

\[ H_1(z) = az^{-2} + bz^{-1} + c \]
\[ H_2(z) = pz^{-2} + qz^{-1} + r \]

Transfer function of the cascaded system

\[ H(z) = H_1(z)H_2(z) \]
\[ = (az^{-2} + bz^{-1} + c)(pz^{-2} + qz^{-1} + r) \]
\[ = apz^{-4} + (aq + bp)z^{-3} + (ar + cp)z^{-2} + (br + qc)z^{-1} + cr \]

So, impulse response \( h[n] \) will be of order 4.

**SOL 6.4.66**

Option (B) is correct.

Output is equal to input with a delay of two units, that is

\[ y(t) = x(t - 2) \]
\[ Y(z) = z^{-2}X(z) \]

Transfer function,

\[ H(z) = \frac{Y(z)}{X(z)} = z^{-2} \]

For the cascaded system, transfer function

\[ H(z) = H_1(z)H_2(z) \]
\[ z^{-2} = \frac{(z - 0.5)}{(z - 0.8)}H_2(z) \]
\[ H_2(z) = \frac{z^{-1} - 0.8z^{-2}}{z - 0.5} = \frac{z^{-2} - 0.8z^{-3}}{1 - 0.5z^{-1}} \]

**SOL 6.4.67**

Option (B) is correct.

\[ y[n] = x[n - 1] \]

or

\[ Y(z) = z^{-1}X(z) \]

or

\[ \frac{Y(z)}{X(z)} = H(z) = z^{-1} \]

Now

\[ H_1(z)H_2(z) = z^{-1} \]
\[ \left( \frac{1 - 0.4z^{-1}}{1 - 0.6z^{-1}} \right)H_2(z) = z^{-1} \]
\[ H_2(z) = \frac{z^{-1}(1 - 0.6z^{-1})}{(1 - 0.4z^{-1})} \]

**SOL 6.4.68**

Option (C) is correct.
We have \( h_1[n] = \delta[n - 1] \) or \( H_1[Z] = Z^{-1} \) and \( h_2[n] = \delta[n - 2] \) or \( H_2(Z) = Z^{-2} \).

Response of cascaded system
\[
H(z) = H_1(z) \cdot H_2(z) = Z^{-1} \cdot Z^{-2} = Z^{-3}
\]

or,
\[
h[n] = \delta[n - 3]
\]

**SOL 6.4.69**

Option (B) is correct.

Let three LTI systems having response \( H_1(z) \), \( H_2(z) \) and \( H_3(z) \) are cascaded as showing below.

\[
\frac{1}{P} \xrightarrow{H_1(z)} \xrightarrow{H_2(z)} \xrightarrow{H_3(z)} H(z)
\]

Assume \( H_1(z) = z^2 + z^1 + 1 \) (non-causal)

\( H_2(z) = z^3 + z^2 + 1 \) (non-causal)

Overall response of the system
\[
H(z) = H_1(z)H_2(z)H_3(z)
\]

\[
= (z^2 + z^1 + 1) (z^3 + z^2 + 1) H_3(z)
\]

To make \( H(z) \) causal we have to take \( H_3(z) \) also causal.

Let
\[
H_3(z) = z^6 + z^4 + 1
\]

\( H(z) = (z^2 + z^1 + 1) (z^3 + z^2 + 1) (z^6 + z^4 + 1) \)

\( H(z) \to \) causal

Similarly to make \( H(z) \) unstable at least one of the system should be unstable.

**SOL 6.4.70**

Option (B) is correct.

\[
H(z) = \frac{1 + az^{-1} + bz^{-2}}{1 + cz^{-1} + dz^{-2} + ez^{-3}}
\]

We know that number of minimum delay elements is equal to the highest power of \( z^{-1} \) present in the denominator of \( H(z) \).

No. of delay elements = 3

**SOL 6.4.71**

Option (A) is correct.

From the given system realization, we can write

\[
(X(z) + Y(z)z^{-2}a_2 + Y(z)a_1z^{-1}) \times a_0 = Y(z)
\]

System Function
\[
H(z) = \frac{Y(z)}{X(z)} = \frac{a_0}{1 - a_1z^{-1} - a_2z^{-2}}
\]

\[
= \frac{1}{a_0} - \frac{a_1}{a_0} z^{-1} - \frac{a_2}{a_0} z^{-2}
\]

Comparing with given \( H(z) \)

\[
\frac{1}{a_0} = 1 \Rightarrow a_0 = 1
\]

\[
-\frac{a_1}{a_0} = -0.7 \Rightarrow a_1 = 0.7
\]

\[
-\frac{a_2}{a_0} = 0.13 \Rightarrow a_2 = -0.13
\]

**SOL 6.4.72**

Option (B) is correct.

Let, \( M \to \) highest power of \( z^{-1} \) in numerator.

\( N \to \) highest power of \( z^{-1} \) in denominator

Number of delay elements in direct form-I realization equals to \( M + N \)
Number of delay elements in direct form-II realization equal to \( N \).
Here, \( M = 3 \), \( N = 3 \)
So delay element in direct form-I realization will be 6 and in direct form realization will be 3.

SOL 6.4.73  Option (A) is correct.
System response is given as
\[
H(z) = \frac{G(z)}{1 - KG(z)}
\]
\( g[n] = \delta[n-1] + \delta[n-2] \)
\( G(z) = z^{-1} + z^{-2} \)
So
\[
H(z) = \frac{(z^{-1} + z^{-2})}{1 - K(z^{-1} + z^{-2})} = \frac{z + 1}{z^2 - Kz - K}
\]
For system to be stable poles should lie inside unit circle.
\[
|z| \leq 1
\]
\[
z = \frac{K \pm \sqrt{K^2 + 4K}}{2} \leq 1
\]
\[
K \pm \sqrt{K^2 + 4K} \leq 2 - K
\]
\[
K^2 + 4K \leq 4 - 4K + K^2
\]
\[
8K \leq 4
\]
\[
K \leq 1/2
\]

SOL 6.4.74  Option (B) is correct.
Input-output relationship of the system
\[
y[n] = x[n] + ay[n-1]
\]
Taking z-transform
\[
Y(z) = X(z) + z^{-1}aY(z)
\]
Transform Function,
\[
\frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}}
\]
Pole of the system \((1 - z^{-1})a = 0 \Rightarrow z = a\)
For stability poles should lie inside the unit circle \(|z| < 1\) so \(|a| < 1\).

SOL 6.4.75  Option (D) is correct.
The relationship between Laplace transform and z-transform is given as
\[
X(s) = X(z)|_{z = e^{st}}
\]
\[
z = e^{st}
\]
\[
\sigma = \text{Re}(s)
\]
\[
\Omega = \text{Im}(s)
\]
We know that
\[
z = re^{i\Omega}
\]
and
\[
s = \sigma + j\omega
\]
From equation (1), (2) and (3), we can write
\[
z = re^{i\Omega} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\Omega t}
\]
From above relation we can find that \(|z| = e^{\sigma t}\) and \(\Omega = \omega T\). It is concluded that,
• If \(\sigma = 0\) then \(|z| = 1\), the \(j\omega\)-axis of s-plane maps into unit circle.
• If \(\sigma < 0\), \(|z| < 1\), it implies that left half of s-plane maps into inside of unit circle (\(|z| < 1\)).
• Similarly, if \(\sigma > 0\), \(|z| > 1\) which implies that right half of s-plane maps into outside of unit circle.
Option (D) is correct.

The relationship between Laplace transform and $z$-transform is given as

$$X(s) = X(z) \big|_{s = e^{st}}$$

$$z = e^{st} \quad \text{...(1)}$$

We know that

$$z = re^{j\omega} \quad \text{...(2)}$$

and

$$s = \sigma + j\omega \quad \text{...(3)}$$

From equation (1), (2) and (3), we can write

$$z = re^{j\omega} = e^{i(j\omega + \sigma)t}$$

$$= e^{t} e^{i\omega t}$$

$$z = re^{j\omega} = e^{t} e^{i\omega t}$$

From above relation we can find that $|z| = e^{|t|}$ and $\omega = \omega T$. It is concluded that,

- If $\sigma = 0$ then $|z| = 1$, the $j\omega$-axis of $s$-plane maps into unit circle.
- If $\sigma < 0$, $|z| < 1$, it implies that left half of $s$-plane maps into inside of unit circle ($|z| < 1$).
- Similarly, if $\sigma > 0$, $|z| > 1$ which implies that right half of $s$-plane maps into outside of unit circle.

Option (C) is correct.

Ideal sampler output is given by

$$f(t) = \sum_{n=0}^{\infty} K_n \delta[t - nT_s]$$

where $T_s \rightarrow$ sampling period

$$n \rightarrow \text{integer}$$

$$f(t) = K_0 \delta[n] + K_1 \delta[n - 1] + K_2 \delta[n - 2] + \ldots$$

$$Z[f(t)] = K_0 + K_1 z^{-1} + K_2 z^{-2} + \ldots + K_n z^{-n}$$

Option (B) is correct.

We know that

$$X(s) = X(z) \big|_{s = e^{st}}$$

$$z = e^{st} \quad \text{so,}$$

$$\ln z = st \quad \text{and}$$

$$s = \frac{\ln z}{T}$$

Option (D) is correct.

Option (C) is correct.

$$H(s) = \frac{a}{s^2 + a^2}$$

Poles in $s$-domain are at $s = \pm ja$. In $z$-domain poles will be at $z = e^{jat}$, so

$$z_1 = e^{-iat} \quad \text{and} \quad z_2 = e^{iat}$$

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